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The periodic unfolding method in domains with holes

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Abstract

We give a comprehensive presentation of the periodic unfolding method for perforated domains, both when the unit hole is a compact subset of the open unit cell and when this is impossible to achieve. In order to apply the method to boundary-value problems with non homogeneous Neumann conditions on the boundaries of the holes, the properties of the boundary unfolding operator are also extensively studied. The paper concludes with applications to such problems and examples of reiterated unfolding.

Keywords: Periodic homogenization, Periodic unfolding, perforated domains, boundary unfolding

Introduction

The periodic unfolding method was introduced in [8] (see also [10]). It gives an elementary proof for the classical periodic homogenization problem, including the case with several micro-scales (a detailed account and proofs can be found in [10]).

It soon became apparent that in the case of periodic problems with holes (of the same size as the period), adapting the method was a way to overcome one of the difficulties of perforated domains. This seemed to be a new point of view. The previous approaches (with the notable exception of [3]) made use of extension operators into the holes (therefore requiring a regularity of these holes and the fact that they are isolated). The unfolding method replaces this by a Poincaré-Wirtinger hypothesis (in a

way similar to that of [3]). As it is, the unfolding operator maps functions defined on the varying perforated domains into functions defined on a fixed domain (albeit of twice the dimension).

The first such use was presented in [13]-[14] and applied to elliptic equations with linear or non linear non homogeneous Robin conditions on the boundaries of the holes. The treatment of non homogeneous conditions on the boundaries of the holes was done by introducing a boundary unfolding operator.

In the spirit of [10], we give a comprehensive presentation of the method for perforated domains. It covers both the case when the unit hole is a compact subset of the open unit cell (as in [13]-[14]), but also that when this is impossible to achieve (this can occur in particular in dimensions larger than 2). The perforated domains we consider here are described in the following manner: a fixed domain Ω is given in \mathbb{R}^n , together with a reference hole S and a basis of the \mathbb{R}^n whose vectors are a generator of the macroscopic periods. The problem is then set in the domain Ω_ε^* , obtained by removing from Ω all the ε -periodic translates of εS . The corresponding unfolding operator $\mathcal{T}_\varepsilon^*$ transforms functions defined on this oscillating domain Ω_ε^* into functions defined on the fixed domain $\Omega \times Y^*$, where Y^* is the reference perforated periodicity cell.

The first main result (see Theorem 3.12) states that if, for every ε , w_ε is a function of $W^{1,p}(\Omega_\varepsilon^*)$ satisfying

$$\|w_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon^*)} \leq C,$$

then, up to a subsequence, there exist w in $W^{1,p}(\Omega)$ and \widehat{w} in $L^p(\Omega; W_{per}^{1,p}(Y^*))$, such that, when ε goes to 0,

$$\mathcal{T}_\varepsilon^*(w_\varepsilon) \rightarrow w \quad \text{strongly in } L_{loc}^p(\Omega; W^{1,p}(Y^*)),$$

$$\mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \widehat{w} \quad \text{weakly in } L^p(\Omega \times Y^*).$$

This compactness result (originally proved in [13]-[14] for isolated holes), is essential for homogenization problems. It requires no extension operator so it does away with the regularity hypothesis on the boundary of Ω_ε^* , necessary for the existence of such extensions. For instance, in the two-dimensional case, the reference hole can be of snow-flake type (see [13] and also [18] and references therein). In this context, when the unit hole is a compact subset of the open reference cell, the condition insuring the existence of extension operators, is replaced by the weaker condition of existence of a Poincaré-Wirtinger inequality in Y^* .

We are also interested in including cases such as Figure 3, where no choice of the basis of periods gives a parallelotop Y satisfying $Y^* \doteq Y \setminus S$ connected, a condition which is necessary for the validity of the Poincaré-Wirtinger inequality in Y^* . In such situations, we show that the method still applies when there exists another reference cell, not necessarily a parallelotop, having the paving property with respect to the period basis and such that the part occupied by the material is connected (see Figure 6).

When S is not compact in Y (the case of non-isolated holes), an extra condition in terms of a Poincaré-Wirtinger inequality is required for the union of the reference cell

and it translates by a period (hypothesis (\mathbf{H}_p) of Section 4). This is similar to the conclusion of Lemma 2.2 of [3], where, however, the boundary of Y^* is assumed to be Lipschitz. We stress the fact that the unfolding approach does not require this extra regularity. It is enough to treat the case of homogeneous Neumann conditions on the boundary of the small holes.

However, in order to treat non-homogeneous Neumann conditions on the boundaries of the holes, the boundary of S has to be assumed Lipschitz (for the surface integrals to make sense). In this case, we give detailed properties of the boundary unfolding operator (which was also originally introduced in [13]-[14]), extending, in particular, the results obtained via two-scale convergence in [2] and [24].

This method (without the boundary unfolding operator) was used (in a preliminary version) for the Stokes problem in [9], for thin piezoelectric perforated shells in [20] and for an elasticity problem related to catalyst supports (which are perforated domains) in [22]. The boundary unfolding method was also used in a different setting for the case of Neumann screens and sieves with perforations of size smaller than ε distributed ε -periodically on a hyperplane ([25] and [11]).

The plan of this paper is as follows:

- Section 1 is a summary of the unfolding method in a fixed domain (without holes);
- Section 2 introduces the notations and basic results for the unfolding method in perforated domains for L^p -functions;
- Section 3 is dedicated to the convergences for sequences of gradients. In Subsection 3.2.1, the geometric condition (\mathbf{H}_p) is introduced under which the separation of scales can be carried out using the macro-micro decomposition. Subsection 3.2.2 deals with the particular case where the periodicity cell is a parallelotop. In the Appendix, we consider the general case, which is more involved due to geometric considerations.
- Section 4 concerns the boundary unfolding operator and its properties.
- In Section 5, the previous results are applied to two new boundary-value problems with non homogeneous Neumann condition on the boundaries of the holes. This gives rise to new terms in the homogenized limit problems (see Theorems 5.6 and 5.13 and Remark 5.14). Finally, we briefly show how these tools are well adapted to the multiscales setting.

The method presented here can be readily adapted to the case of small ε -periodic cracks. This will be presented in a forthcoming publication (see also [17]).

General notations.

In this work, ε indicates the generic element of a bounded subset of \mathbb{R}_+^* in the closure

of which 0 lies. Convergence of ε to 0 is understood in this set. Also, c and C denote generic constants which do not depend upon ε .

As usual, 1_D denotes the characteristic function of the set D .

For a measurable set D in \mathbb{R}^n , $|D|$ denotes its Lebesgue measure.

For simplicity, the notation $L^p(\mathcal{O})$ will be used both for scalar and vector-valued functions defined on the set \mathcal{O} , since no ambiguity will arise.

1 A brief summary of the unfolding method in fixed domains

Let $\mathbf{b} = (b_1, \dots, b_n)$ be a basis in \mathbb{R}^n . Set

$$\mathcal{G} = \left\{ \xi \in \mathbb{R}^n \mid \xi = \sum_{i=1}^n k_i b_i, (k_1, \dots, k_n) \in \mathbb{Z}^n \right\}. \quad (1.1)$$

By Y , we denote the open parallelotop generated by the basis \mathbf{b} , i.e.,

$$\left\{ y \in \mathbb{R}^n \mid y = \sum_{i=1}^n y_i b_i, (y_1, \dots, y_n) \in (0, 1)^n \right\}. \quad (1.2)$$

In periodic homogenization, Y is often called the reference cell and \mathbf{b} is the set of the reference periods. However, the latter notation is better suited for \mathcal{G} itself, since any other free generator set of \mathcal{G} can be used in place of \mathbf{b} .

For $z \in \mathbb{R}^n$, $[z]_Y$ denotes the unique (up to a set of measure zero) integer combination $\sum_{j=1}^n k_j b_j$ of the periods such that $z - [z]_Y$ belongs to Y (see Figure 1). Set now

$$\{z\}_Y = z - [z]_Y \in Y \quad \text{a.e. for } z \in \mathbb{R}^n.$$

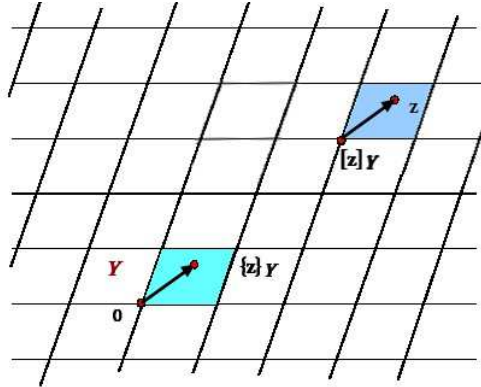


Figure 1. Definition of $[z]_Y$ and $\{z\}_Y$

In particular, for positive ε , one has

$$x = \varepsilon \left(\left[\frac{x}{\varepsilon} \right]_Y + \left\{ \frac{x}{\varepsilon} \right\}_Y \right) \quad \text{for all } x \in \mathbb{R}^n.$$

Let now Ω be an open subset (not necessarily bounded) of \mathbb{R}^n . We recall the notations used in [10],

$$\Xi_\varepsilon = \left\{ \xi \in \mathcal{G}, \varepsilon(\xi + Y) \subset \Omega \right\}, \quad (1.3)$$

$$\widehat{\Omega}_\varepsilon = \text{interior} \left\{ \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \overline{Y}) \right\}, \quad \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon. \quad (1.4)$$

The set $\widehat{\Omega}_\varepsilon$ is the interior of the largest union of $\varepsilon(\xi + \overline{Y})$ cells ($\xi \in \mathcal{G}$), such that $\varepsilon(\xi + Y)$ are included in Ω , while Λ_ε is the subset of Ω containing the parts from $\varepsilon(\xi + \overline{Y})$ cells intersecting the boundary $\partial\Omega$ (see Figure 2).

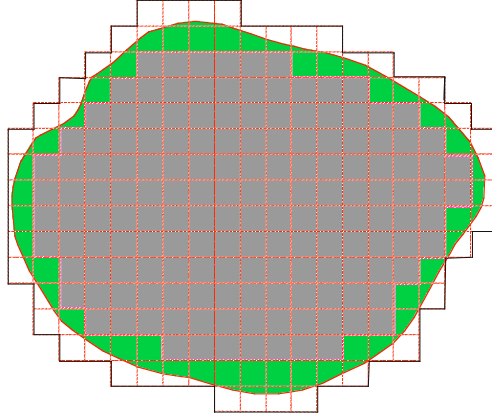


Figure 2. The sets $\widehat{\Omega}_\varepsilon$ (in grey) and Λ_ε (in green)

By construction, Λ_ε is a kind of boundary layer near $\partial\Omega$ and its limit is the empty set, so that for every bounded open set $\omega \subset \mathbb{R}^n$, the Lebesgue measure $|\Lambda_\varepsilon \cap \omega|$ converges to 0.

Remark 1.1. *The theory developed in [10] can easily be applied with $\widehat{\Omega}_\varepsilon$ replaced by any subdomain $\widehat{\Omega}'_\varepsilon$ of Ω which is a union of ε -cells included in Ω and which converges to Ω , i.e.,*

$$\begin{aligned} \widehat{\Omega}'_\varepsilon &= \text{interior} \left\{ \bigcup_{\xi \in \Xi'_\varepsilon} \varepsilon(\xi + \overline{Y}) \right\} \quad \text{with } \Xi'_\varepsilon \subset \Xi_\varepsilon, \\ \sup_{x \in \partial\widehat{\Omega}'_\varepsilon} \text{dist}(x, \partial\Omega) &\rightarrow 0. \end{aligned} \quad (1.5)$$

As a matter of fact, all the results in [10] are still true for any choice of $\widehat{\Omega}'_\varepsilon$ satisfying (1.5) (except for the quantitative corrector results which require $\sup_{x \in \partial\widehat{\Omega}'_\varepsilon} \text{dist}(x, \partial\Omega) = O(\varepsilon)$). For the study of perforated domains in the Appendix, we will have to use such a generalized setup.

We start by recalling several results from [10], essential in periodic homogenization problems. For other properties and related comments, we refer the reader to [8] and [10].

Definition 1.2. For ϕ Lebesgue-measurable on Ω , the unfolding operator \mathcal{T}_ε is defined as follows:

$$\mathcal{T}_\varepsilon(\phi)(x, y) = \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{a.e. for } (x, y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & \text{a.e. for } (x, y) \in \Lambda_\varepsilon \times Y. \end{cases} \quad (1.6)$$

Note that if ϕ is Lebesgue-measurable on $\widehat{\Omega}_\varepsilon$, then for ϕ extended by zero in $\Omega \setminus \widehat{\Omega}_\varepsilon$, definition (1.6) makes sense. The operator \mathcal{T}_ε maps functions defined on Ω into functions defined on the domain $\Omega \times Y$ (which are piece-wise constant with respect to x , more precisely constant on each ε -cell of $\widehat{\Omega}_\varepsilon$).

Theorem 1.3. Let p belong to $[1, +\infty)$.

- (i) \mathcal{T}_ε is linear continuous from $L^p(\Omega)$ to $L^p(\Omega \times Y)$. Its norm is bounded by $|Y|^{1/p}$.
- (ii) Let $\{w_\varepsilon\}$ be a sequence in $L^p(\Omega)$ such that

$$w_\varepsilon \rightarrow w \quad \text{strongly in } L^p(\Omega).$$

Then

$$\mathcal{T}_\varepsilon(w_\varepsilon) \rightarrow w \quad \text{strongly in } L^p(\Omega \times Y).$$

- (iii) Let $\{w_\varepsilon\}$ be bounded in $L^p(\Omega)$ and suppose that the corresponding $\mathcal{T}_\varepsilon(w_\varepsilon)$ (which is bounded in $L^p(\Omega \times Y)$) converges weakly to \widehat{w} in $L^p(\Omega \times Y)$. Then

$$w_\varepsilon \rightharpoonup \mathcal{M}_Y(\widehat{w}) = \frac{1}{|Y|} \int_Y \widehat{w}(\cdot, y) dy \quad \text{weakly in } L^p(\Omega).$$

Theorem 1.4. Let $\{w_\varepsilon\}$ be in $W^{1,p}(\Omega)$ with $p \in (1, +\infty)$, and assume that $\{w_\varepsilon\}$ is a bounded sequence in $W^{1,p}(\Omega)$. Then, there exist a subsequence (still denoted ε), w in $W^{1,p}(\Omega)$ and \widehat{w} in $L^p(\Omega; W_{per}^{1,p}(Y))$, such that

$$\begin{aligned} \mathcal{T}_\varepsilon(w_\varepsilon) &\rightharpoonup w \quad \text{weakly in } L^p(\Omega; W^{1,p}(Y)), \\ \mathcal{T}_\varepsilon(\nabla w_\varepsilon) &\rightharpoonup \nabla w + \nabla_y \widehat{w} \quad \text{weakly in } L^p(\Omega \times Y). \end{aligned} \quad (1.7)$$

Here, $W_{per}^{1,p}(Y)$ denotes the space of the functions in $W_{loc}^{1,p}(\mathbb{R}^n)$ which are G -periodic. It is a closed subspace of $W^{1,p}(Y)$ and is endowed with the corresponding norm.

We end this section by recalling the notion of averaging operator \mathcal{U}_ε . This operator is the adjoint of \mathcal{T}_ε and maps $L^p(\Omega \times Y)$ into the space $L^p(\Omega)$.

Definition 1.5. For p in $[1, +\infty]$, the averaging operator $\mathcal{U}_\varepsilon : L^p(\Omega \times Y) \mapsto L^p(\Omega)$ is defined as follows:

$$\mathcal{U}_\varepsilon(\Phi)(x) = \begin{cases} \frac{1}{|Y|} \int_Y \Phi\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}_Y\right) dz & \text{a.e. for } x \in \widehat{\Omega}_\varepsilon, \\ 0 & \text{a.e. for } x \in \Lambda_\varepsilon. \end{cases}$$

For $\psi \in L^p(\Omega)$ and $\Phi \in L^{p'}(\Omega \times Y)$, one has

$$\int_\Omega \mathcal{U}_\varepsilon(\Phi)(x) \psi(x) dx = \frac{1}{|Y|} \int_{\Omega \times Y} \Phi(x, y) \mathcal{T}_\varepsilon(\psi)(x, y) dx dy. \quad (1.8)$$

The operator \mathcal{U}_ε is almost a left-inverse of \mathcal{T}_ε , as for ϕ in $L^p(\Omega)$

$$\mathcal{U}_\varepsilon(\mathcal{T}_\varepsilon(\phi))(x) = \begin{cases} \phi(x) & \text{a.e. for } x \in \widehat{\Omega}_\varepsilon, \\ 0 & \text{a.e. for } x \in \Lambda_\varepsilon. \end{cases} \quad (1.9)$$

It is not a right-inverse, since for Φ in $L^p(\Omega \times Y)$,

$$\mathcal{T}_\varepsilon(\mathcal{U}_\varepsilon(\Phi))(x, y) = \begin{cases} \frac{1}{|Y|} \int_Y \Phi\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, y\right) dz, & \text{a.e. for } (x, y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & \text{a.e. for } (x, y) \in \Lambda_\varepsilon \times Y. \end{cases} \quad (1.10)$$

The main properties of \mathcal{U}_ε are listed in the next proposition.

Proposition 1.6. (Properties of \mathcal{U}_ε). Suppose that p is in $[1, +\infty)$.

(i) The averaging operator is linear and continuous from $L^p(\Omega \times Y)$ to $L^p(\Omega)$ and

$$\|\mathcal{U}_\varepsilon(\Phi)\|_{L^p(\Omega)} \leq |Y|^{-1/p} \|\Phi\|_{L^p(\Omega \times Y)}.$$

(ii) If φ is independent of y and belongs to $L^p(\Omega)$, then

$$\mathcal{U}_\varepsilon(\varphi) \rightarrow \varphi \quad \text{strongly in } L^p(\Omega).$$

(iii) Let $\{\Phi_\varepsilon\}$ be a bounded sequence in $L^p(\Omega \times Y)$ such that $\Phi_\varepsilon \rightharpoonup \Phi$ weakly in $L^p(\Omega \times Y)$. Then

$$\mathcal{U}_\varepsilon(\Phi_\varepsilon) \rightharpoonup \mathcal{M}_Y(\Phi) = \frac{1}{|Y|} \int_Y \Phi(\cdot, y) dy \quad \text{weakly in } L^p(\Omega).$$

In particular, for every $\Phi \in L^p(\Omega \times Y)$,

$$\mathcal{U}_\varepsilon(\Phi) \rightharpoonup \mathcal{M}_Y(\Phi) \quad \text{weakly in } L^p(\Omega).$$

(iv) Suppose that $\{w_\varepsilon\}$ is a sequence in $L^p(\Omega)$. Then, the following assertions are equivalent:

$$(a) \quad \mathcal{T}_\varepsilon(w_\varepsilon) \rightarrow \widehat{w} \quad \text{strongly in } L^p(\Omega \times Y) \quad \text{and} \quad \int_{\Lambda_\varepsilon} |w_\varepsilon|^p dx \rightarrow 0,$$

$$(b) \quad w_\varepsilon - \mathcal{U}_\varepsilon(\widehat{w}) \rightarrow 0 \quad \text{strongly in } L^p(\Omega).$$

We complete this section with a somewhat unusual convergence property involving the averaging operator \mathcal{U}_ε and which is applied in Theorem 5.9.

Proposition 1.7. *For $p \in [1, +\infty)$, suppose that α is in $L^p(\Omega)$ and β in $L^\infty(\Omega; L^p(Y))$. Then, the product $\mathcal{U}_\varepsilon(\alpha)\mathcal{U}_\varepsilon(\beta)$ belongs to $L^p(\Omega)$ and*

$$\mathcal{U}_\varepsilon(\alpha\beta) - \mathcal{U}_\varepsilon(\alpha)\mathcal{U}_\varepsilon(\beta) \rightarrow 0 \text{ strongly in } L^p(\Omega). \quad (1.11)$$

Proof. It suffices to prove (1.11) and to do so, consider first the function $\mathcal{U}_\varepsilon(\alpha)\mathcal{U}_\varepsilon(\beta)$ in the set $\varepsilon(\xi + Y) \times Y$, $\xi \in \Xi_\varepsilon$. Using the fact that on this set $\mathcal{U}_\varepsilon(\alpha)$ is constant on the cell $\varepsilon(\xi + Y)$, one has $\mathcal{U}_\varepsilon(\alpha)\mathcal{U}_\varepsilon(\beta) = \mathcal{U}_\varepsilon(\mathcal{U}_\varepsilon(\alpha)\beta)$. Therefore,

$$\int_{\varepsilon(\xi+Y)} |\mathcal{U}_\varepsilon(\alpha\beta) - \mathcal{U}_\varepsilon(\alpha)\mathcal{U}_\varepsilon(\beta)|^p dx = \int_{\varepsilon(\xi+Y)} |\mathcal{U}_\varepsilon([\alpha - \mathcal{U}_\varepsilon(\alpha)]\beta)|^p dx.$$

Summing over $\xi \in \Xi_\varepsilon$ gives

$$\begin{aligned} \|\mathcal{U}_\varepsilon(\alpha\beta) - \mathcal{U}_\varepsilon(\alpha)\mathcal{U}_\varepsilon(\beta)\|_{L^p(\Omega)}^p &= \int_{\Omega} |\mathcal{U}_\varepsilon([\alpha - \mathcal{U}_\varepsilon(\alpha)]\beta)|^p dx \\ &\leq \frac{1}{|Y|} \|(\alpha - \mathcal{U}_\varepsilon(\alpha))\beta\|_{L^p(\Omega \times Y)}^p \leq \frac{1}{|Y|} \|\alpha - \mathcal{U}_\varepsilon(\alpha)\|_{L^p(\Omega)}^p \|\beta\|_{L^\infty(\Omega; L^p(Y))}^p. \end{aligned}$$

The last expression goes to 0 by Proposition 1.6 (ii). \square

2 The unfolding method in perforated domains: the case of L^p -functions

2.1 Definition and notations for perforated domains

In periodic homogenization, considering a problem posed in a fixed domain (with no perforations), one can always choose the reference cell Y as a parallelotop (with one vertex at the origin). It is dramatically different in presence of holes, where more complex situations can occur.

In Section 3.2, we will see that in the case of periodically perforated domains, it is essential that the perforated reference cell be connected. But for the domain depicted in Figure 3, it is impossible to choose the reference cell Y as a parallelotop so that it remains connected after perforation. Similar examples can be given in higher dimensions. The presentation we give here is able to deal with such cases.

We use a setting which is well-adapted to perforated domains and for which results similar to those of Section 1 still hold. It is based on the notion of a domain **having the paving property with respect to the group \mathbf{G}** in \mathbb{R}^n (up to null sets, it is the same as the notion of fundamental domain under the action of the group \mathbf{G}).

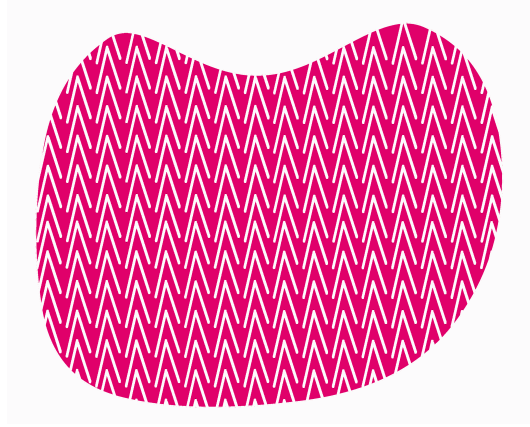


Figure 3. An example of “strangely” perforated domain in two dimensions

Definition 2.1. *The bounded open set Y has the paving property with respect to the group G when it is connected, its boundary ∂Y is the null set and*

$$\mathbb{R}^n = \bigcup_{\xi \in \mathcal{G}} (\xi + \overline{Y}), \quad \forall (\xi_1, \xi_2) \in \mathcal{G}^2, \xi_1 \neq \xi_2 \implies (\xi_1 + Y) \cap (\xi_2 + Y) = \emptyset. \quad (2.1)$$

Observe that any translate of Y satisfies the same property and that εY has the same paving property as Y but with \mathcal{G} replaced by $\varepsilon \mathcal{G}$.

From now on, we reserve the notation Y (the reference cell) for a bounded open set having the paving property with respect to the group \mathcal{G} . We use the notation \mathcal{P} for the open parallelotop generated by the basis \mathbf{b} , which was denoted Y in section 1, i.e.,

$$\mathcal{P} = \left\{ y \in \mathbb{R}^n \mid y = \sum_{i=1}^n y_i b_i, (y_1, \dots, y_n) \in (0, 1)^n \right\}. \quad (2.2)$$

The parallelotop \mathcal{P} plays an important role, in particular in the Appendix for the definition of the macro-micro operators $\mathcal{Q}_\varepsilon^*$ and $\mathcal{R}_\varepsilon^*$ (the analogues of \mathcal{Q}_ε and \mathcal{R}_ε of [10]).

Remark 2.2. *The parallelotop \mathcal{P} has the same paving property as Y . Furthermore, the spaces $W_{per}^{1,p}(\mathcal{P})$ and $W_{per}^{1,p}(Y)$ are the same. Indeed, they are both obtained by restricting to \mathcal{P} and to Y , the elements of $W_{loc}^{1,p}(\mathbb{R}^n)$ which are invariant under the action of \mathcal{G} . These functions have the same traces on the opposite faces of \mathcal{P} (or Y).*

Now, let S be a closed strict subset of \overline{Y} and denote by Y^* the part occupied by the material i.e. $Y^* = Y \setminus S$ (see Figure 4, where Y happens to be a parallelotop, but the definition holds for a general Y). From now on, the sets S and Y^* will be the reference hole and perforated cell respectively.

Concerning the domain of Figure 3, where no choice of parallelotop gives a connected Y^* , there are many possible Y 's that give a connected Y^* . An example in dimension 2 is given in Figure 5. Similar situations can occur in higher dimensions.

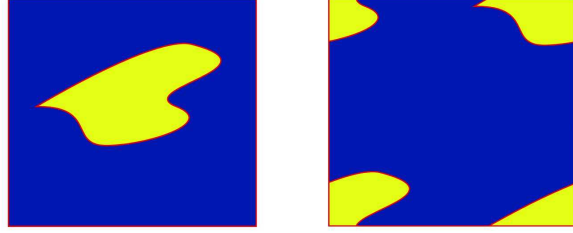


Figure 4. Two versions of $Y^* = Y \setminus S$ (in blue) for the same perforations

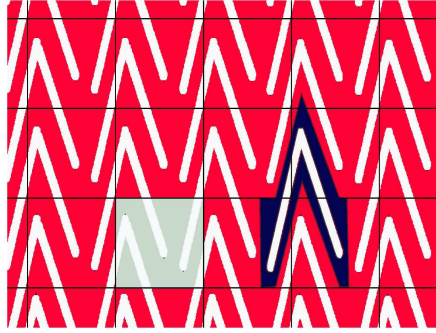


Figure 5. An example of Y^* (in dark blue); \mathcal{P} in light beige.

Let now Ω be a given domain in \mathbb{R}^n . The perforated domain Ω_ε^* is obtained by removing from Ω the set of holes S_ε (see Figure 5 for the two-dimensional case),

$$\Omega_\varepsilon^* = \Omega \setminus S_\varepsilon, \quad \text{where } S_\varepsilon = \bigcup_{\xi \in \mathcal{G}} \varepsilon(\xi + S). \quad (2.3)$$

The following notations will be used (see Figure 6):

$$\widehat{\Omega}_\varepsilon^* = \widehat{\Omega}_\varepsilon \setminus S_\varepsilon, \quad \Lambda_\varepsilon^* = \Omega_\varepsilon^* \setminus \widehat{\Omega}_\varepsilon^*, \quad \widehat{\partial S}_\varepsilon = \partial \widehat{\Omega}_\varepsilon^* \cap \partial S_\varepsilon, \quad (2.4)$$

where the set $\widehat{\Omega}_\varepsilon$ is defined, as before, by (1.4). The boundary of the set of holes in Ω is $\partial S_\varepsilon \cap \Omega$ while $\widehat{\partial S}_\varepsilon$ denotes the boundary of the holes that are included in $\widehat{\Omega}_\varepsilon$.

We will also use similar notations when applied to the whole of \mathbb{R}^n ,

$$\begin{aligned} (\mathbb{R}^n)^* &= \mathbb{R}^n \setminus \bigcup_{\xi \in \mathcal{G}} (\xi + \overline{S}), \\ (\mathbb{R}^n)_\varepsilon^* &= \mathbb{R}^n \setminus \overline{S_\varepsilon}. \end{aligned} \quad (2.5)$$

By this definition, $(\mathbb{R}^n)^*$ is nothing else than \mathbb{R}^n perforated G-periodically by S , while $(\mathbb{R}^n)_\varepsilon^*$ is the \mathbb{R}^n perforated by ε G-periodically by εS . Consequently, another equivalent definition for Ω_ε^* is

$$\Omega_\varepsilon^* = (\mathbb{R}^n)_\varepsilon^* \cap \Omega.$$

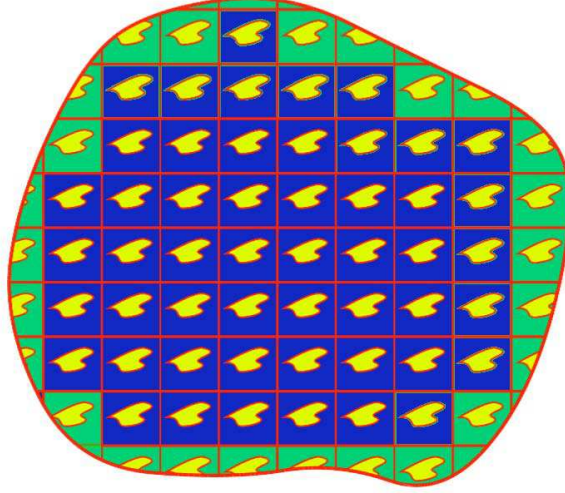


Figure 6. The sets Ω_ε^* , $\hat{\Omega}_\varepsilon^*$ (in dark blue) and Λ_ε^* (in light green)

2.2 The unfolding operator $\mathcal{T}_\varepsilon^*$ in perforated domains

In this subsection, we define an unfolding operator $\mathcal{T}_\varepsilon^*$ specific to perforated domains, following the ideas of the preceding section (see in particular, Definition 1.2). The first characteristic of this operator is that it maps functions defined on the oscillating domain Ω_ε^* to functions defined on the fixed domain $\Omega \times Y^*$.

The definitions used here differ slightly from those introduced originally in [13]-[14]. They follow the usage of [10] for fixed domains (as recalled in Section 1). This allows to treat more general situations (such as in Section 5). The proofs of [13]-[14] carry over in the present setting. For the sake of completeness, these proofs are included here.

Notation

1. Extensions by zero. If a function g is defined on a set $\mathcal{O} \setminus \mathcal{A}$, its extension by zero in \mathcal{A} will be denoted either by \tilde{g} , or by $[g]^\sim$.

2. Mean value. For any measurable set \mathcal{O} of finite measure, $\mathcal{M}_\mathcal{O}$ denotes the mean value over the set \mathcal{O} , i.e.,

$$\mathcal{M}_\mathcal{O}(\Phi)(\cdot) = \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} \Phi(\cdot, y) dy, \quad \forall \Phi \in L^1(\Omega \times \mathcal{O}). \quad (2.6)$$

Remark 2.3. With the above notations, it follows from (2.6) that

$$\mathcal{M}_{Y^*}(\Phi)(\cdot) = \frac{|Y|}{|Y^*|} \mathcal{M}_Y(\tilde{\Phi})(\cdot), \quad \forall \Phi \in L^1(\Omega \times Y^*).$$

Definition 2.4. For any function ϕ Lebesgue-measurable on Ω_ε^* , the unfolding operator $\mathcal{T}_\varepsilon^*$ is defined by

$$\mathcal{T}_\varepsilon^*(\phi)(x, y) = \begin{cases} \phi\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{a.e. for } (x, y) \in \widehat{\Omega}_\varepsilon \times Y^*, \\ 0 & \text{a.e. for } (x, y) \in \Lambda_\varepsilon \times Y^*. \end{cases} \quad (2.7)$$

Obviously, for $v, w \in L^p(\Omega_\varepsilon^*)$, $\mathcal{T}_\varepsilon^*(v w) = \mathcal{T}_\varepsilon^*(v) \mathcal{T}_\varepsilon^*(w)$. For ϕ Lebesgue-measurable on $\widehat{\Omega}_\varepsilon^*$, we extend it by zero in $\Omega_\varepsilon^* \setminus \widehat{\Omega}_\varepsilon^*$, so the above definition makes sense.

Remark 2.5. The relationship between \mathcal{T}_ε and $\mathcal{T}_\varepsilon^*$ is given for any w defined on Ω_ε^* , by

$$\mathcal{T}_\varepsilon^*(w) = \mathcal{T}_\varepsilon(\tilde{w})|_{\Omega \times Y^*}. \quad (2.8)$$

Actually, the previous equality still holds with every extension of w from Ω_ε^* into Ω . In particular, for w defined on Ω ,

$$\mathcal{T}_\varepsilon^*(w|_{\Omega_\varepsilon^*}) = \mathcal{T}_\varepsilon(w)|_{\Omega \times Y^*}. \quad (2.9)$$

Because of relationship (2.8), the operator $\mathcal{T}_\varepsilon^*$ enjoys properties which follow directly from those of \mathcal{T}_ε listed in Theorems 1.3 and 1.4.

Proposition 2.6. For $p \in [1, +\infty)$, the operator $\mathcal{T}_\varepsilon^*$ is linear and continuous from $L^p(\Omega_\varepsilon^*)$ to $L^p(\Omega \times Y^*)$. For every ϕ in $L^1(\Omega_\varepsilon^*)$ and w in $L^p(\Omega_\varepsilon^*)$

$$\begin{aligned} (i) \quad & \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi)(x, y) \, dx \, dy = \int_{\widehat{\Omega}_\varepsilon^*} \phi(x) \, dx = \int_{\Omega_\varepsilon^*} \phi(x) \, dx - \int_{\Lambda_\varepsilon^*} \phi(x) \, dx, \\ (ii) \quad & \|\mathcal{T}_\varepsilon^*(w)\|_{L^p(\Omega \times Y^*)} = |Y|^{1/p} \|w\|_{L^p(\Omega_\varepsilon^*)} \leq |Y|^{1/p} \|w\|_{L^p(\Omega_\varepsilon^*)} \end{aligned}$$

.

Corollary 2.7. Let ϕ_ε be in $L^1(\Omega_\varepsilon^*)$ and satisfying

$$\int_{\Lambda_\varepsilon^*} |\phi_\varepsilon| \, dx \rightarrow 0. \quad (2.10)$$

Then

$$\int_{\Omega_\varepsilon^*} \phi_\varepsilon \, dx - \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon) \, dx \, dy \rightarrow 0.$$

As a consequence of Remark 2.3, Remark 2.5 and Theorem 1.3 the following results hold.

Proposition 2.8. *Let p belong to $[1, +\infty)$.*

(i) *For $w \in L^p(\Omega)$,*

$$\mathcal{T}_\varepsilon^*(w) \rightarrow w \quad \text{strongly in } L^p(\Omega \times Y^*).$$

(ii) *Let $w_\varepsilon \in L^p(\Omega_\varepsilon^*)$ such that $\|w_\varepsilon\|_{L^p(\Omega_\varepsilon^*)} \leq C$. If*

$$\mathcal{T}_\varepsilon^*(w_\varepsilon) \rightharpoonup \widehat{w} \quad \text{weakly in } L^p(\Omega \times Y^*),$$

then

$$\widetilde{w}_\varepsilon \rightharpoonup \frac{|Y^*|}{|Y|} \mathcal{M}_{Y^*}(\widehat{w}) \quad \text{weakly in } L^p(\Omega).$$

Remark 2.9. *This last result (statement (ii) above) implies that for w_ε in $L^p(\Omega_\varepsilon^*)$ such that $\|w_\varepsilon\|_{L^p(\Omega_\varepsilon^*)}$ is bounded, the following are equivalent:*

a) *There is $w \in L^p(\Omega)$ such that*

$$\widetilde{w}_\varepsilon \rightharpoonup \frac{|Y^*|}{|Y|} w \quad \text{weakly in } L^p(\Omega).$$

b) *All the weak limit points W in $L^p(\Omega \times Y^*)$ of the sequence $\{\mathcal{T}_\varepsilon^*(w_\varepsilon)\}$ have the same average over Y^* (this average $\mathcal{M}_{Y^*}(W)$ being just w).*

2.3 The averaging operator $\mathcal{U}_\varepsilon^*$ in perforated domains

We now determine the adjoint of $\mathcal{T}_\varepsilon^*$. To do so, let v be in $L^p(\Omega \times Y^*)$ and u in $L^{p'}(\Omega_\varepsilon^*)$. Then, by (1.8) and (2.8),

$$\begin{aligned} \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(u)(x, y) v(x, y) dx dy &= \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon(\widetilde{u})(x, y) v(x, y) dx dy \\ &= \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(\widetilde{u})(x, y) \widetilde{v}(x, y) dx dy \\ &= \int_{\Omega} \widetilde{u}(x) \mathcal{U}_\varepsilon(\widetilde{v})(x) dx = \int_{\Omega_\varepsilon^*} u(x) \mathcal{U}_\varepsilon(\widetilde{v})(x) dx. \end{aligned}$$

This gives the formula for the averaging operator $\mathcal{U}_\varepsilon^*$,

$$\mathcal{U}_\varepsilon^*(v) = \mathcal{U}_\varepsilon(\widetilde{v})|_{\Omega_\varepsilon^*}, \quad (2.11)$$

hence the following definition.

Definition 2.10. *For p in $[1, +\infty]$, the averaging operator $\mathcal{U}_\varepsilon^* : L^p(\Omega \times Y^*) \mapsto L^p(\Omega_\varepsilon^*)$ is defined as*

$$\mathcal{U}_\varepsilon^*(\Phi)(x) = \begin{cases} \frac{1}{|Y|} \int_Y \Phi\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}_Y\right) dz & \text{a.e. for } x \in \widehat{\Omega}_\varepsilon^*, \\ 0 & \text{a.e. for } x \in \Lambda_\varepsilon^*. \end{cases}$$

Note that if Φ belongs to $L^p(\Omega \times Y)$, then $\mathcal{U}_\varepsilon^*(\Phi|_{\Omega \times Y^*}) = \mathcal{U}_\varepsilon(\Phi)|_{\Omega_\varepsilon^*}$.

Proposition 2.11. *Let p be in $[1, +\infty)$. Then, for any φ in $L^p(\Omega)$,*

$$\|\varphi - \mathcal{U}_\varepsilon^*(\varphi)\|_{L^p(\Omega_\varepsilon^*)} \rightarrow 0. \quad (2.12)$$

Proof. Since $\mathcal{U}_\varepsilon^*(\phi) = \mathcal{U}_\varepsilon(\phi)|_{\Omega_\varepsilon^*}$, (2.12) is immediate from Proposition 1.6 (ii). \square

As a consequence of (2.11), we get

Proposition 2.12. *Let p belong to $[1, +\infty]$. The averaging operator is linear and continuous from $L^p(\Omega \times Y^*)$ to $L^p(\Omega_\varepsilon^*)$ and*

$$\|\mathcal{U}_\varepsilon^*(\Phi)\|_{L^p(\Omega_\varepsilon^*)} \leq |Y|^{-1/p} \|\Phi\|_{L^p(\Omega \times Y^*)}.$$

From (2.8) and (2.11) it follows that $\mathcal{U}_\varepsilon^*$ is almost a left-inverse of $\mathcal{T}_\varepsilon^*$. Indeed, for every ϕ in $L^p(\Omega_\varepsilon^*)$ one has

$$\mathcal{U}_\varepsilon^*(\mathcal{T}_\varepsilon^*(\phi))(x) = \begin{cases} \phi(x) & \text{a.e. for } x \in \widehat{\Omega}_\varepsilon^*, \\ 0 & \text{a.e. for } x \in \Lambda_\varepsilon^*, \end{cases} \quad (2.13)$$

while, for every Φ in $L^p(\Omega \times Y^*)$,

$$\mathcal{T}_\varepsilon^*(\mathcal{U}_\varepsilon^*(\Phi))(x, y) = \begin{cases} \frac{1}{|Y|} \int_Y \Phi\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, y\right) dz & \text{a.e. for } (x, y) \in \widehat{\Omega}_\varepsilon \times Y^*, \\ 0 & \text{a.e. for } (x, y) \in \Lambda_\varepsilon \times Y^*. \end{cases} \quad (2.14)$$

Proposition 2.13. *(Properties of $\mathcal{U}_\varepsilon^*$). Suppose that p is in $[1, +\infty)$.*

(i) *Let $\{\Phi_\varepsilon\}$ be a bounded sequence in $L^p(\Omega \times Y^*)$ such that $\Phi_\varepsilon \rightharpoonup \Phi$ weakly in $L^p(\Omega \times Y^*)$. Then*

$$\widetilde{\mathcal{U}}_\varepsilon^*(\Phi_\varepsilon) \rightharpoonup \frac{|Y^*|}{|Y|} \mathcal{M}_{Y^*}(\Phi) \quad \text{weakly in } L^p(\Omega).$$

In particular, for every $\Phi \in L^p(\Omega \times Y^)$,*

$$\widetilde{\mathcal{U}}_\varepsilon^*(\Phi) \rightharpoonup \frac{|Y^*|}{|Y|} \mathcal{M}_{Y^*}(\Phi) \quad \text{weakly in } L^p(\Omega),$$

(contrary to the case without holes, this convergence is never strong for $\Phi \not\equiv 0$, because of the oscillations of $\widehat{\Omega}_\varepsilon$).

(ii) *Let $\{\Phi_\varepsilon\}$ be a sequence such that $\Phi_\varepsilon \rightarrow \Phi$ strongly in $L^p(\Omega \times Y^*)$. Then*

$$\mathcal{T}_\varepsilon^*(\mathcal{U}_\varepsilon^*(\Phi_\varepsilon)) \rightarrow \Phi \quad \text{strongly in } L^p(\Omega \times Y^*).$$

(iii) *Let w_ε be in $L^p(\Omega_\varepsilon^*)$. Then, the following assertions are equivalent:*

$$(a) \quad \mathcal{T}_\varepsilon^*(w_\varepsilon) \rightarrow \widehat{w} \quad \text{strongly in } L^p(\Omega \times Y^*),$$

$$(b) \quad \|w_\varepsilon - \mathcal{U}_\varepsilon^*(\widehat{w})\|_{L^p(\widehat{\Omega}_\varepsilon^*)} \rightarrow 0.$$

(iv) Let w_ε be in $L^p(\Omega_\varepsilon^*)$. Then, the following assertions are equivalent:

$$(c) \quad \mathcal{T}_\varepsilon^*(w_\varepsilon) \rightarrow \widehat{w} \quad \text{strongly in } L^p(\Omega \times Y^*) \quad \text{and} \quad \int_{\Lambda_\varepsilon^*} |w_\varepsilon|^p dx \rightarrow 0,$$

$$(d) \quad \|w_\varepsilon - \mathcal{U}_\varepsilon^*(\widehat{w})\|_{L^p(\Omega_\varepsilon^*)} \rightarrow 0.$$

Proof. Using (2.11) this proposition is a transcription of Proposition 1.6. \square

Remark 2.14. Assertions (iii)(b) and (iv)(d) are corrector-type results.

Corollary 2.15. Let w_ε be in $L^p(\Omega_\varepsilon^*)$ and w in $L^p(\Omega)$. Then the following assertions are equivalent:

$$(a) \quad \mathcal{T}_\varepsilon^*(w_\varepsilon) \rightarrow w \quad \text{strongly in } L^p(\Omega \times Y^*),$$

$$(b) \quad \|w_\varepsilon - w\|_{L^p(\widehat{\Omega}_\varepsilon^*)} \rightarrow 0.$$

Furthermore, (a) together with $\int_{\Lambda_\varepsilon^*} |w_\varepsilon|^p dx \rightarrow 0$, is equivalent to $\|w_\varepsilon - w\|_{L^p(\Omega_\varepsilon^*)} \rightarrow 0$.

Proof. The first result follows from the inequality

$$\left| \|w_\varepsilon - w\|_{L^p(\widehat{\Omega}_\varepsilon^*)} - \|w_\varepsilon - \mathcal{U}_\varepsilon^*(w)\|_{L^p(\widehat{\Omega}_\varepsilon^*)} \right| \leq \|w - \mathcal{U}_\varepsilon^*(w)\|_{L^p(\Omega_\varepsilon^*)},$$

together with Proposition 2.11 and Proposition 2.13 (iii). The second equivalence follows from the first one and Proposition 2.13 (iv). \square

3 The unfolding method in perforated domains: the case of $W^{1,p}$ -functions

In this section, we consider sequences $\{w_\varepsilon\}$ such that for each ε , w_ε belongs to $W^{1,p}(\Omega_\varepsilon^*)$. Two cases are considered.

3.1 The first case: $\|w_\varepsilon\|_{L^p(\Omega_\varepsilon^*)} + \varepsilon \|\nabla w_\varepsilon\|_{L^p(\Omega_\varepsilon^*)}$ bounded

The main result of this subsection is the following theorem:

Theorem 3.1. *Let p be in $(1, +\infty]$. Let w_ε belong to $W^{1,p}(\Omega_\varepsilon^*)$ and satisfy*

$$\|w_\varepsilon\|_{L^p(\Omega_\varepsilon^*)} + \varepsilon \|\nabla w_\varepsilon\|_{L^p(\Omega_\varepsilon^*)} \leq C. \quad (3.1)$$

Then, there exists some $\widehat{w} \in L^p(\Omega; W_{per}^{1,p}(Y^))$, such that, up to a subsequence,*

$$\begin{aligned} \mathcal{T}_\varepsilon^*(w_\varepsilon) &\rightharpoonup \widehat{w} \quad \text{weakly in } L^p(\Omega; W^{1,p}(Y^*)), \\ \varepsilon \mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) &\rightharpoonup \nabla_y \widehat{w} \quad \text{weakly in } L^p(\Omega \times Y^*). \end{aligned} \quad (3.2)$$

The delicate point in the proof is the Y -periodicity of \widehat{w} . If the reference cell Y is a parallelotop and $\bar{S} \subset Y$, one can simply argue as in the case without holes comparing the traces on opposite faces of Y (see [10]). However, in the general case, Y is not a parallelotop, the boundary of S may not be Lipschitz or the way ∂S intersects ∂Y can be such that traces are not meaningful.

To circumvent this difficulty, we use an approach which avoids the use of traces: we introduce an auxiliary bigger cell \mathcal{Y} (the union of 2^n contiguous copies of Y) and use a new operator $\mathcal{T}_\varepsilon^{\mathcal{Y}*}$. We stress the fact that this operator is not the unfolding operator corresponding to unit cell \mathcal{Y} , but is a “natural” way to extend $\mathcal{T}_\varepsilon^*$ to the set $\Omega \times \mathcal{Y}^*$.

The formal definition of \mathcal{Y} uses the notations

$$\begin{aligned} \mathcal{K} &= \text{the set of vertices of } \mathcal{P} = \left\{ \ell \in \mathbb{R}^n \mid \ell = \sum_{I=1}^n k_I b_I, (k_1, \dots, k_n) \in \{0, 1\}^n \right\}, \\ \mathcal{Y} &= \text{interior} \left\{ \bigcup_{\ell \in \mathcal{K}} (\ell + \bar{Y}) \right\}. \end{aligned} \quad (3.3)$$

The corresponding sets $\widehat{\Omega}_\varepsilon^{\mathcal{Y}}$ and $\Lambda_\varepsilon^{\mathcal{Y}}$ are defined as before,

$$\Xi_\varepsilon^{\mathcal{Y}} = \left\{ \xi \in \Xi_\varepsilon \mid \varepsilon(\xi + \bar{\mathcal{Y}}) \subset \Omega \right\}. \quad (3.4)$$

$$\widehat{\Omega}_\varepsilon^{\mathcal{Y}} = \text{interior} \left\{ \bigcup_{\xi \in \Xi_\varepsilon^{\mathcal{Y}}} \varepsilon(\xi + \bar{Y}) \right\}, \quad \Lambda_\varepsilon^{\mathcal{Y}} = \Omega \setminus \widehat{\Omega}_\varepsilon^{\mathcal{Y}}, \quad (3.5)$$

This definition implies that if ξ belongs to $\Xi_\varepsilon^{\mathcal{Y}}$ then for all $\ell \in \mathcal{K}$, $\xi + \ell \in \Xi_\varepsilon$. Consequently, if a cell $\varepsilon(\xi + Y)$ is included in $\widehat{\Omega}_\varepsilon^{\mathcal{Y}}$, then, all its translates $\varepsilon(\xi + \ell + Y)$ with $\ell \in \mathcal{K}$ are in $\widehat{\Omega}_\varepsilon$ (see Figure 7).

We also have the notation

$$\mathcal{Y}^* = \mathcal{Y} \cap (\mathbb{R}^n)^*. \quad (3.6)$$

For ϕ Lebesgue-measurable on Ω_ε^* , $\mathcal{T}_\varepsilon^{\mathcal{Y}*}(\phi)$ is defined as follows:

$$\mathcal{T}_\varepsilon^{\mathcal{Y}*}(\phi)(x, y) = \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{a.e. for } (x, y) \in \widehat{\Omega}_\varepsilon^{\mathcal{Y}} \times \mathcal{Y}^*, \\ 0 & \text{a.e. for } (x, y) \in \Lambda_\varepsilon^{\mathcal{Y}} \times \mathcal{Y}^*. \end{cases} \quad (3.7)$$

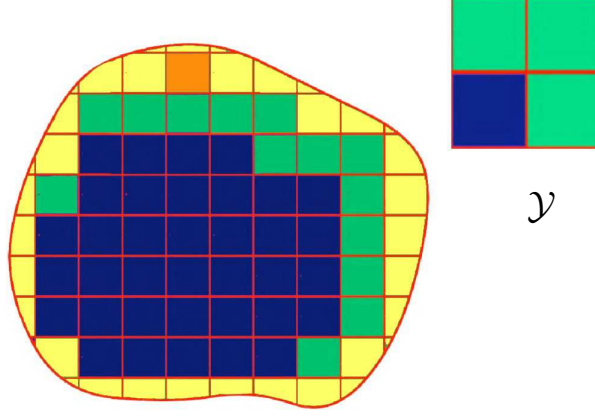


Figure 7. The set $\widehat{\Omega}_\varepsilon^\mathcal{Y}$ (in dark blue) and \mathcal{Y} (in dark blue and green)

Due to the definition of $\widehat{\Omega}_\varepsilon^\mathcal{Y}$ given in (3.5), formula (3.7) makes sense. Note here the use of $\left[\frac{x}{\varepsilon}\right]_Y$ and not of $\left[\frac{x}{\varepsilon}\right]_\mathcal{Y}$.

From (3.7), it is easy to check that if w belongs to $L^p(\Omega_\varepsilon^*)$, then $\mathcal{T}_\varepsilon^{\mathcal{Y}*}(w)$ is in the space $L^p(\Omega \times \mathcal{Y}^*)$ and

$$\|\mathcal{T}_\varepsilon^{\mathcal{Y}*}(w)\|_{L^p(\Omega \times \mathcal{Y}^*)} \leq (2^n |Y|)^{1/p} \|w\|_{L^p(\Omega_\varepsilon^*)}. \quad (3.8)$$

The main properties relating $\mathcal{T}_\varepsilon^{\mathcal{Y}*}$ with $\mathcal{T}_\varepsilon^*$ are that for $x \in \widehat{\Omega}_\varepsilon^\mathcal{Y}$ and every vector of the basis $\mathbf{b} = (b_1, \dots, b_n)$

$$\begin{aligned} \mathcal{T}_\varepsilon^{\mathcal{Y}*}(\phi)(x, y) &= \mathcal{T}_\varepsilon^*(\phi)(x, y) \quad \text{a.e. for } y \in Y^*, \\ \mathcal{T}_\varepsilon^{\mathcal{Y}*}(\phi)(x, y + b_k) &= \mathcal{T}_\varepsilon^*(\phi)(x + \varepsilon b_k, y) \quad \text{a.e. for } y \in Y^*. \end{aligned} \quad (3.9)$$

In particular, if ω is a relatively compact open subset of Ω , then for ε sufficiently small,

$$\mathcal{T}_\varepsilon^{\mathcal{Y}*}(\phi)(x, y + b_k) = \mathcal{T}_\varepsilon^*(\phi)(x + \varepsilon b_k, y) \quad \text{a.e. on } \omega \times Y^*. \quad (3.10)$$

Proof of Theorem 3.1. Consider a function w in $W^{1,p}(\Omega_\varepsilon^*)$. As in the case without holes, it is straightforward that

$$\nabla_y(\mathcal{T}_\varepsilon^*(w)) = \varepsilon \mathcal{T}_\varepsilon^*(\nabla w), \quad \text{a.e. for } (x, y) \in \Omega \times Y^*. \quad (3.11)$$

This implies that $\mathcal{T}_\varepsilon^*$ maps $W^{1,p}(\Omega_\varepsilon^*)$ into $L^p(\Omega; W^{1,p}(Y^*))$. Obviously, a similar property is also true for $\mathcal{T}_\varepsilon^{\mathcal{Y}*}$ with \mathcal{Y}^* instead of Y^* .

Using ii) of Proposition 2.6, (3.11) and (3.1) it follows that $\{\mathcal{T}_\varepsilon^*(w_\varepsilon)\}$ is bounded in $L^p(\Omega; W^{1,p}(Y^*))$, so that convergences (3.2) hold (up to a subsequence).

It remains to prove that \widehat{w} is periodic. To do so, consider the unfolded function $\mathcal{T}_\varepsilon^{\mathcal{Y}^*}(w_\varepsilon)$. By the same argument as just above, there exist a sequence (still denoted ε) and \bar{w} in $L^p(\Omega; W^{1,p}(\mathcal{Y}^*))$, such that

$$\mathcal{T}_\varepsilon^{\mathcal{Y}^*}(w_\varepsilon) \rightharpoonup \bar{w} \quad \text{weakly in } L^p(\Omega; W^{1,p}(\mathcal{Y}^*)), \quad (3.12)$$

(or weak-* for $p = +\infty$).

Let ω be an open bounded set whose closure is included in Ω . From (3.9) and (3.2) it follows that

$$\bar{w}(x, y) = \widehat{w}(x, y) \quad \text{for a.e. } (x, y) \in \omega \times Y^*.$$

Now, let Φ be in $\mathcal{D}(\omega \times Y^*)$. From (3.10) for ε small enough and for every k in $\{1, \dots, n\}$, one has

$$\begin{aligned} \int_{\omega \times Y^*} \mathcal{T}_\varepsilon^{\mathcal{Y}^*}(w_\varepsilon)(x, y + b_k) \Phi(x, y) \, dx \, dy &= \int_{\omega \times Y^*} \mathcal{T}_\varepsilon^*(w_\varepsilon)(x + \varepsilon b_k, y) \Phi(x, y) \, dx \, dy \\ &= \int_{\omega \times Y^*} \mathcal{T}_\varepsilon^*(w_\varepsilon)(x, y) \Phi(x - \varepsilon b_k, y) \, dx \, dy. \end{aligned}$$

Passing to the limit gives

$$\int_{\omega \times Y^*} \bar{w}(x, y + b_k) \Phi(x, y) \, dx \, dy = \int_{\omega \times Y^*} \widehat{w}(x, y) \Phi(x, y) \, dx \, dy.$$

and using (3.2) and (3.12) gives

$$\bar{w}(x, y + b_k) = \widehat{w}(x, y) \quad \text{for a.e. } (x, y) \in \omega \times Y^*.$$

Since this holds for every $\omega \subset\subset \Omega$, it is also true in $\Omega \times Y^*$.

Therefore, \bar{w} is actually \mathbf{b} -periodic and can be extended by periodicity to the whole of $(\mathbb{R}^n)^*$. This shows that \widehat{w} is the restriction of \bar{w} to Y^* , proving that it belongs to $L^p(\Omega; W_{per}^{1,p}(Y^*))$. \square

Remark 3.2. Let p be in $(1, +\infty]$, and some κ in $\{1, \dots, n\}$. Suppose that w_ε belongs to $L^p(\Omega_\varepsilon^*)$ and its gradient is bounded only in the direction of a period, i.e. satisfies

$$\|w_\varepsilon\|_{L^p(\Omega_\varepsilon^*)} + \varepsilon \|\nabla w_\varepsilon \cdot b_\kappa\|_{L^p(\Omega_\varepsilon^*)} \leq C. \quad (3.13)$$

Then, by similar arguments there exist a sequence (still denoted ε) and \widehat{w} in $L^p(\Omega \times Y^*)$ with $\nabla_y \widehat{w} \cdot b_\kappa$ in $L^p(\Omega \times Y^*)$, such that

$$\mathcal{T}_\varepsilon^*(w_\varepsilon) \rightharpoonup \widehat{w} \quad \text{weakly in } L^p(\Omega \times Y^*), \quad (3.14)$$

$$\varepsilon \mathcal{T}_\varepsilon^*(\nabla w_\varepsilon \cdot b_\kappa) = \nabla_y(\mathcal{T}_\varepsilon^*(w_\varepsilon)) \cdot b_\kappa \rightharpoonup \nabla_y \widehat{w} \cdot b_\kappa \quad \text{weakly in } L^p(\Omega \times Y^*),$$

weak-* for $p = +\infty$. Moreover, the limit function \widehat{w} is b_κ -periodic.

3.2 The second case: $\|w_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon^*)}$ or $\|\nabla w_\varepsilon\|_{L^p(\Omega_\varepsilon^*)}$ bounded

We consider the space $W^{1,p}(\Omega_\varepsilon^*)$ or, for a given Γ_0 open subset of $\partial\Omega$, its subspace denoted by $W_0^{1,p}(\Omega_\varepsilon^*; \Gamma_0 \cap \partial\Omega_\varepsilon^*)$ of functions vanishing on $\Gamma_0 \cap \partial\Omega_\varepsilon^*$. For such a case, we assume that Ω has a Lipschitz boundary. If Γ_0 is not empty, this regularity of $\partial\Omega$ implies that for every Ω' open subset of \mathbb{R}^n such that $\Omega \subset \Omega'$ and $\Gamma_0 = \partial\Omega \cap \Omega'$,

$$W_0^{1,p}(\Omega_\varepsilon^*; \Gamma_0 \cap \partial\Omega_\varepsilon^*) = \left\{ \phi \in W^{1,p}(\Omega_\varepsilon^*) \mid \exists \phi' \in W^{1,p}((\Omega')_\varepsilon^*) \right. \\ \left. \phi' = 0 \text{ in } (\Omega')_\varepsilon^* \setminus \overline{\Omega_\varepsilon^*} \text{ and } \phi = \phi'|_{\Omega_\varepsilon^*} \right\}. \quad (3.15)$$

In order to use the macro-micro decomposition as in the case without holes, a geometric condition is needed here, which will be expressed in terms of the Poincaré-Wirtinger inequality.

3.2.1 The Poincaré-Wirtinger inequality: hypothesis ($\mathbf{H_p}$)

Definition 3.3. *A bounded open set \mathcal{O} satisfies the Poincaré-Wirtinger inequality for the exponent $p \in [1, +\infty]$ if there exists a constant C_p such that*

$$\forall u \in W^{1,p}(\mathcal{O}), \quad \|u - \mathcal{M}_\mathcal{O}(u)\|_{L^p(\mathcal{O})} \leq C_p \|\nabla u\|_{L^p(\mathcal{O})}.$$

Obviously, for \mathcal{O} to satisfy the above condition, it has to be connected. Conversely, there are extra conditions for this property to hold (e.g. for John domains, see [18]). The simplest such condition is that the boundary of \mathcal{O} be Lipschitz (by the compactness of the Rellich theorem), but it is far from necessary.

Remark 3.4. *It is also known that if two bounded open sets \mathcal{O}_1 and \mathcal{O}_2 satisfy the Poincaré-Wirtinger inequality with the same exponent p , then $\mathcal{O} \doteq \text{interior}(\overline{\mathcal{O}_1 \cup \mathcal{O}_2})$ satisfies Definition 3.3 with the same exponent p if and only if \mathcal{O} is connected.*

We can now state the geometric condition.

The **Geometrical hypothesis ($\mathbf{H_p}$)** is satisfied when the following hold: *the open set Y^* satisfies the Poincaré-Wirtinger inequality for the exponent p ($p \in [1, +\infty]$) and for every vector b_i , $i \in \{1, \dots, n\}$, of the basis of G , the interior of $Y^* \cup (b_i + Y^*)$ is connected.*

Note that under hypothesis ($\mathbf{H_p}$) and owing to Remark 3.4, the open set \mathcal{Y}^* defined by (3.6), satisfies the Poincaré-Wirtinger inequality for the same exponent p as Y^* .

Should $\partial Y^* \cap \partial Y$ be Lipschitz – so traces exist – one could use the approach of Section 3 of [10] with the relative compactness of the sequence of local averages obtained via the Kolmogorov criterion (as in the proof of Lemma 2.3 of [3]), and the Y -periodicity of the function \widehat{w} by comparing traces on opposite faces of Y^* .

Here, we do not assume such regularity, allowing in particular ∂S and ∂Y to intersect in an arbitrary fashion. We therefore use the approach of Section 4 of [10], i.e. the macro-micro decomposition. To simplify the presentation, we consider first the case where Y

is a parallelotop. The general case, where Y cannot be chosen as the parallelotop \mathcal{P} , exhibits extra geometric complexities and is presented in the Appendix.

3.2.2 The macro-micro operators $\mathcal{Q}_\varepsilon^*$ and $\mathcal{R}_\varepsilon^*$ when the reference cell Y is a parallelotop

As in the case without holes, the macro approximation will be defined by an average at the points of Ξ_ε (see (1.3)) and extended to the set $\widehat{\Omega}_\varepsilon^\mathcal{Y}$ by Q_1 -interpolation (continuous and piece-wise polynomials of degree ≤ 1 with respect to each coordinate). The notations $\mathcal{Y}, \mathcal{K}, \dots$, are those of subsection 3.1.

Definition 3.5. *The operator $\mathcal{Q}_\varepsilon^* : L^p(\Omega_\varepsilon^*) \mapsto W^{1,\infty}(\widehat{\Omega}_\varepsilon^\mathcal{Y})$, for $p \in [1, +\infty]$, is defined as follows:*

$$\mathcal{Q}_\varepsilon^*(\phi)(\varepsilon\xi) = \frac{1}{|Y^*|} \int_{Y^*} \phi(\varepsilon\xi + \varepsilon z) dz = \mathcal{M}_{\varepsilon\xi + \varepsilon Y^*}(\phi) \quad \text{for all } \xi \text{ in } \Xi_\varepsilon^\mathcal{Y} + \mathcal{K},$$

and for every $x \in \widehat{\Omega}_\varepsilon^\mathcal{Y}$,

$$\begin{cases} \mathcal{Q}_\varepsilon^*(\phi)(x) \text{ is the } Q_1\text{-interpolate of the values of } \mathcal{Q}_\varepsilon^*(\phi) \text{ at the vertices} \\ \text{of the cell } \varepsilon \left[\frac{x}{\varepsilon} \right]_Y + \varepsilon Y. \end{cases} \quad (3.16)$$

Proposition 3.6. *(Properties of $\mathcal{Q}_\varepsilon^*$) For ϕ in $L^p(\Omega_\varepsilon^*)$, $p \in [1, +\infty]$, there exists constants C depending on n , Y and Y^* only, such that*

$$\begin{aligned} (i) \quad & \|\mathcal{Q}_\varepsilon^*(\phi)\|_{L^\infty(\widehat{\Omega}_\varepsilon^\mathcal{Y})} \leq \frac{C}{\varepsilon^{n/p}} \|\phi\|_{L^p(\Omega_\varepsilon^*)}, \\ (ii) \quad & \|\mathcal{Q}_\varepsilon^*(\phi)\|_{L^p(\widehat{\Omega}_\varepsilon^\mathcal{Y})} \leq C \|\phi\|_{L^p(\Omega_\varepsilon^*)}. \end{aligned} \quad (3.17)$$

Proof. By definition, the Q_1 -interpolate of a function ϕ defined on the vertices of Y is a polynomial with degree less or equal to one with respect to each variable, hence it is Lipschitz-continuous and reaches its maximum at some vertex. So, to estimate the L^∞ -norm of $\mathcal{Q}_\varepsilon^*(\phi)$ in the cell $\varepsilon(\xi + Y)$ it suffices to estimate the maximum of the $|\mathcal{Q}_\varepsilon^*(\phi)(\varepsilon\xi)|$. Indeed, by Jensen's inequality,

$$|\mathcal{Q}_\varepsilon^*(\phi)(\varepsilon\xi)|^p \leq \frac{1}{|Y^*|} \int_{Y^*} |\phi(\varepsilon\xi + \varepsilon z)|^p dz = \frac{1}{\varepsilon^n |Y^*|} \int_{\varepsilon\xi + \varepsilon Y^*} |\phi(x)|^p dx, \quad (3.18)$$

and on the other hand

$$\frac{1}{\varepsilon^n |Y^*|} \int_{\varepsilon\xi + \varepsilon Y^*} |\phi(x)|^p dx \leq \frac{1}{\varepsilon^n |Y^*|} \|\phi\|_{L^p(\Omega_\varepsilon^*)}^p.$$

Estimate (i) then follows with $C = \frac{1}{|Y^*|^{1/p}}$.

The space $Q_1(Y)$ of all Q_1 interpolates defined on Y is of dimension 2^n ; therefore all its norms are equivalent. Thus, there is a constant c such that for every $\Phi \in Q_1(Y)$,

$$\|\Phi\|_{L^p(Y)} \leq c \left(\sum_{\ell \in \mathcal{K}} |\Phi(\ell)|^p \right)^{1/p}.$$

Rescaling this inequality for $\Phi(y) \doteq \mathcal{Q}_\varepsilon^*(\phi)(\varepsilon\xi + \varepsilon y)$, gives

$$\|\mathcal{Q}_\varepsilon^*(\phi)\|_{L^p(\varepsilon\xi + \varepsilon Y)} \leq c\varepsilon^{n/p} \left(\sum_{\ell \in \mathcal{K}} \left| \mathcal{Q}_\varepsilon^*(\phi)(\varepsilon\xi + \varepsilon\ell) \right|^p \right)^{1/p}.$$

Using (3.18) we immediately get (ii) by summation over $\Xi_\varepsilon^{\mathcal{Y}}$ (after taking the p -th power). \square

In the remainder of this section we assume that hypothesis (H_p) holds.

The following proposition is well-known from the Finite Elements Method.

Proposition 3.7. *There is a constant C independent of ε such that for every $\phi \in W^{1,p}(\Omega_\varepsilon^*)$,*

$$\|\nabla \mathcal{Q}_\varepsilon^*(\phi)\|_{L^p(\hat{\Omega}_\varepsilon^{\mathcal{Y}})} \leq C \|\nabla \phi\|_{L^p(\Omega_\varepsilon^*)}.$$

Proof. By a similar argument as above, but for the gradient of every function in the space $Q_1(Y)$, there is a constant c such that

$$\|\nabla \Phi\|_{L^p(Y)} \leq c \left(\sum_{\ell \in \mathcal{K}} |\Phi(\ell) - \Phi(0)|^p \right)^{1/p}.$$

Rescaling this inequality for $\Phi(y) \doteq \mathcal{Q}_\varepsilon^*(\phi)(\varepsilon\xi + \varepsilon y)$, gives

$$\|\nabla \mathcal{Q}_\varepsilon^*(\phi)\|_{L^p(\varepsilon\xi + \varepsilon Y)} \leq c\varepsilon^{(1-\frac{p}{n})} \left(\sum_{\ell \in \mathcal{K}} \left| \mathcal{Q}_\varepsilon^*(\phi)(\varepsilon\xi + \varepsilon\ell) - \mathcal{Q}_\varepsilon^*(\phi)(\varepsilon\xi) \right|^p \right)^{1/p}. \quad (3.19)$$

For $\psi \in W^{1,p}(\mathcal{Y}^*)$, apply the Poincaré-Wirtinger inequality in the domain \mathcal{Y}^* to obtain

$$\|\psi - \mathcal{M}_{\mathcal{Y}^*}(\psi)\|_{L^p(\mathcal{Y}^*)} \leq C \|\nabla \psi\|_{L^p(\mathcal{Y}^*)}. \quad (3.20)$$

Integrating $\psi - \mathcal{M}_{\mathcal{Y}^*}(\psi)$ over Y^* and $Y^* + \ell$ for ℓ in \mathcal{K} , and using the above inequality gives respectively

$$|\mathcal{M}_{Y^*}(\psi) - \mathcal{M}_{\mathcal{Y}^*}(\psi)| \leq \frac{1}{|Y^*|^{1/p}} \|\psi - \mathcal{M}_{\mathcal{Y}^*}(\psi)\|_{L^p(\mathcal{Y}^*)} \leq C \|\nabla \psi\|_{L^p(\mathcal{Y}^*)},$$

and

$$|\mathcal{M}_{Y^*+\ell}(\psi) - \mathcal{M}_{Y^*}(\psi)| \leq \frac{1}{|Y^*|^{1/p}} \|\psi - \mathcal{M}_{Y^*}(\psi)\|_{L^p(Y^*)} \leq C \|\nabla \psi\|_{L^p(Y^*)}.$$

The constant C depends on the Poincaré-Wirtinger constant in (3.20) and on $|Y^*|^{1/p}$. Therefore, for every ℓ in \mathcal{K} ,

$$|\mathcal{M}_{Y^*+\ell}(\psi) - \mathcal{M}_{Y^*}(\psi)| \leq C \|\nabla \psi\|_{L^p(Y^*)}. \quad (3.21)$$

For ϕ in $W^{1,p}(\Omega_\varepsilon^*)$ and every $\xi \in \Xi_\varepsilon^\mathcal{Y}$, a scaling argument gives

$$\left| \mathcal{Q}_\varepsilon^*(\phi)(\varepsilon\xi + \varepsilon\ell) - \mathcal{Q}_\varepsilon^*(\phi)(\varepsilon\xi) \right| \leq \varepsilon^{\frac{p}{n}} C \|\nabla \phi\|_{L^p(\varepsilon\xi + \varepsilon Y^*)},$$

which, in combination with (3.19) yields

$$\|\nabla \mathcal{Q}_\varepsilon^*(\phi)\|_{L^p(\varepsilon\xi + \varepsilon Y^*)} \leq \varepsilon C \|\nabla \phi\|_{L^p(\varepsilon\xi + \varepsilon Y^*)}.$$

Proposition 3.7 follows by summation over $\Xi_\varepsilon^\mathcal{Y}$. \square

Now, every function ϕ in $L^p(\Omega_\varepsilon^*)$ can be decomposed on the set $\widehat{\Omega}_\varepsilon^{**} \doteq \Omega_\varepsilon^* \cap \widehat{\Omega}_\varepsilon^\mathcal{Y}$ into the sum of two terms as follows:

$$\phi = \mathcal{Q}_\varepsilon^*(\phi) + \mathcal{R}_\varepsilon^*(\phi), \quad \text{a.e. in } \widehat{\Omega}_\varepsilon^{**} \doteq \Omega_\varepsilon^* \cap \widehat{\Omega}_\varepsilon^\mathcal{Y}. \quad (3.22)$$

Proposition 3.8. *There is a constant C independent of ε such that for every ϕ in $W^{1,p}(\Omega_\varepsilon^*)$,*

$$\begin{aligned} (i) \quad & \|\mathcal{R}_\varepsilon^*(\phi)\|_{L^p(\widehat{\Omega}_\varepsilon^{**})} \leq \varepsilon C \|\nabla \phi\|_{L^p(\Omega_\varepsilon^*)}, \\ (ii) \quad & \|\nabla \mathcal{R}_\varepsilon^*(\phi)\|_{L^p(\widehat{\Omega}_\varepsilon^{**})} \leq C \|\nabla \phi\|_{L^p(\Omega_\varepsilon^*)}. \end{aligned}$$

Proof. The second estimate is a simple consequence of Proposition 3.7.

To prove inequality (i), consider first some $\psi \in W^{1,p}(Y^*)$, and apply the Poincaré-Wirtinger inequality in Y^* to obtain

$$\|\psi - \mathcal{M}_{Y^*}(\psi)\|_{L^p(Y^*)} \leq C \|\nabla \psi\|_{L^p(Y^*)}. \quad (3.23)$$

For ϕ in $W^{1,p}(\Omega_\varepsilon^*)$ and every $\xi \in \Xi_\varepsilon^\mathcal{Y}$, from (3.23) by a scaling argument it follows that

$$\|\phi - \mathcal{Q}_\varepsilon^*(\phi)(\varepsilon\xi)\|_{L^p(\varepsilon\xi + \varepsilon Y^*)} \leq C\varepsilon \|\nabla \phi\|_{L^p(\varepsilon\xi + \varepsilon Y^*)}. \quad (3.24)$$

On the other hand, using (3.20) and the definition of $\mathcal{Q}_\varepsilon^*(\phi)$, a similar scaling argument gives

$$\|\mathcal{Q}_\varepsilon^*(\phi) - \mathcal{Q}_\varepsilon^*(\phi)(\varepsilon\xi)\|_{L^p(\varepsilon\xi + \varepsilon Y^*)} \leq C\varepsilon \|\nabla \phi\|_{L^p(\varepsilon\xi + \varepsilon Y^*)}.$$

The last two inequalities combine into

$$\|\mathcal{R}_\varepsilon^*(\phi)\|_{L^p(\varepsilon\xi + \varepsilon Y^*)} \leq C\varepsilon \|\nabla \phi\|_{L^p(\varepsilon\xi + \varepsilon Y^*)},$$

from which inequality (i) follows by summation over $\Xi_\varepsilon^\mathcal{Y}$. \square

As a corollary of Proposition 3.8, we obtain the following uniform Poincaré inequality which is useful for applications. It is based on the micro-macro decomposition (carried out here on a neighborhood of Ω , which we may take to be the whole of \mathbb{R}^n). It extends to the case of non-smooth holes the result of Lemma A4 of [3].

Theorem 3.9. *Assume that Ω is bounded in one direction and with Lipschitz boundary. Then there exists a constant C independent of ε such that*

$$\forall \phi \in W_0^{1,p}(\Omega_\varepsilon^*; \partial\Omega \cap \partial\Omega_\varepsilon^*), \quad \|\phi\|_{L^p(\Omega_\varepsilon^*)} \leq C \|\nabla\phi\|_{L^p(\Omega_\varepsilon^*)}. \quad (3.25)$$

Proof. We extend ϕ by zero to the whole of $(\mathbb{R}^n)_\varepsilon^*$ (see (2.5)), extension still denoted ϕ . Then, for the macro and micro operators associated with \mathbb{R}^n , Propositions 3.7 and 3.8 apply and give

$$\|\nabla\mathcal{Q}_\varepsilon^*(\phi)\|_{L^p(\mathbb{R}^n)} + \frac{1}{\varepsilon}\|\mathcal{R}_\varepsilon^*(\phi)\|_{L^p((\mathbb{R}^n)_\varepsilon^*)} \leq C \|\nabla\phi\|_{L^p(\Omega_\varepsilon^*)}.$$

The domain Ω being bounded in one direction, the support of $\mathcal{Q}_\varepsilon^*(\phi)$ is contained in the 1-neighborhood of Ω , denoted $\mathcal{V}_1(\Omega)$ (for ε small enough!) and bounded in the same direction. Hence, using the usual Poincaré inequality in $\mathcal{V}_1(\Omega)$, we have

$$\|\mathcal{Q}_\varepsilon^*(\phi)\|_{L^p(\mathbb{R}^n)} \leq C \|\nabla\mathcal{Q}_\varepsilon^*(\phi)\|_{L^p(\mathbb{R}^n)} \leq C \|\nabla\phi\|_{L^p(\Omega_\varepsilon^*)},$$

which gives (3.25), since $\phi = \mathcal{Q}_\varepsilon^*(\phi) + \mathcal{R}_\varepsilon^*(\phi)$, a.e. in $(\mathbb{R}^n)_\varepsilon^*$. \square

3.2.3 Convergence results

We suppose that p is in $(1, +\infty)$, that Hypothesis (\mathbf{H}_p) holds and consider sequences such that

$$\|w_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon^*)} \leq C \quad \text{or} \quad \|\nabla w_\varepsilon\|_{L^p(\Omega_\varepsilon^*)} \leq C.$$

The main convergence results are Theorems 3.12 and 3.13 below.

Notation. For every $w \in L^p(\Omega_\varepsilon^*)$, the function $\mathcal{Q}_\varepsilon^*(w)$ is defined on $\widehat{\Omega}_\varepsilon^\mathcal{V}$ (Subsection 3.2.2) or on $\widehat{\Omega}_\varepsilon^\mathcal{P}$ (Appendix). Its extension by 0 to \mathbb{R}^n is $\widetilde{\mathcal{Q}_\varepsilon^*(w_\varepsilon)}$ which is then restricted to Ω . For simplicity, this restriction is still denoted $\widetilde{\mathcal{Q}_\varepsilon^*(w_\varepsilon)}$. Similarly, the function $\mathcal{R}_\varepsilon^*(w)$, defined on $\widehat{\Omega}_\varepsilon^{**}$ is extended by 0 to $(\mathbb{R}^n)_\varepsilon^*$ and then restricted to Ω_ε^* , this restriction is still denoted $\widetilde{\mathcal{R}_\varepsilon^*(w_\varepsilon)}$. Also, we denote by $[\nabla\mathcal{Q}_\varepsilon^*(w_\varepsilon)]^\sim$ (resp. $[\nabla\mathcal{R}_\varepsilon^*(w_\varepsilon)]^\sim$) the extension by 0 to \mathbb{R}^n (or Ω) of $\nabla\mathcal{Q}_\varepsilon^*(w_\varepsilon)$ (resp. $\nabla\mathcal{R}_\varepsilon^*(w_\varepsilon)$).

We first prove a convergence result concerning the sequence $\{\mathcal{Q}_\varepsilon^*(w_\varepsilon)\}$. In Lemma 3.10 and Theorem 3.12, the set Γ_0 is a non-empty open subset of $\partial\Omega$. Since $\widetilde{\mathcal{Q}_\varepsilon^*(w_\varepsilon)}$ and $[\nabla\mathcal{Q}_\varepsilon^*(w_\varepsilon)]^\sim$ are defined on Ω , the original unfolding operator \mathcal{T}_ε is used.

Lemma 3.10. *Let w_ε be in $W^{1,p}(\Omega_\varepsilon^*)$ satisfying*

$$\|w_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon^*)} \leq C. \quad (3.26)$$

Then, there exists a function w in $W^{1,p}(\Omega)$ such that, up to a subsequence,

$$\begin{aligned} (i) \quad & \mathcal{T}_\varepsilon(\widetilde{\mathcal{Q}}_\varepsilon^*(w_\varepsilon)) \rightarrow w \quad \text{strongly in } L_{loc}^p(\Omega; W^{1,p}(Y)), \\ (ii) \quad & \mathcal{T}_\varepsilon(\widetilde{\mathcal{Q}}_\varepsilon^*(w_\varepsilon)) \rightharpoonup w \quad \text{weakly in } L^p(\Omega; W^{1,p}(Y)), \\ (iii) \quad & \mathcal{T}_\varepsilon([\nabla \mathcal{Q}_\varepsilon^*(w_\varepsilon)]^\sim) \rightharpoonup \nabla w \quad \text{weakly in } L^p(\Omega \times Y). \end{aligned} \quad (3.27)$$

Moreover, assuming that Ω is with Lipschitz boundary, if for every ε the function w_ε belongs to $W_0^{1,p}(\Omega_\varepsilon^*; \Gamma_0 \cap \partial\Omega)$, then w belongs to $W_0^{1,p}(\Omega; \Gamma_0)$ (i.e. the trace of w vanishes on Γ_0).

Proof. All the convergences of this proof are up to a subsequence. From Propositions 3.6 (ii) and Proposition 3.7, both sequences $\{\widetilde{\mathcal{Q}}_\varepsilon^*(w_\varepsilon)\}$ and $\{[\nabla \mathcal{Q}_\varepsilon^*(w_\varepsilon)]^\sim\}$ are bounded in $L^p(\Omega)$. Then there exist w in $L^p(\Omega)$ and $F \in L^p(\Omega)$ such that

$$\begin{aligned} (i) \quad & \widetilde{\mathcal{Q}}_\varepsilon^*(w_\varepsilon) \rightarrow w \quad \text{strongly in } L_{loc}^p(\Omega), \\ (ii) \quad & \widetilde{\mathcal{Q}}_\varepsilon^*(w_\varepsilon) \rightharpoonup w \quad \text{weakly in } L^p(\Omega), \\ (iii) \quad & [\nabla \mathcal{Q}_\varepsilon^*(w_\varepsilon)]^\sim \rightharpoonup F \quad \text{weakly in } L^p(\Omega). \end{aligned} \quad (3.28)$$

Let ω be a relatively compact open subset in Ω and let Φ be in $(\mathcal{D}(\omega))^n$. For ε sufficiently small, the inclusions $\text{support}(\Phi) \subset \omega \subset \widehat{\Omega}_\varepsilon^\gamma$ hold. Therefore,

$$\begin{aligned} \int_\Omega [\nabla \mathcal{Q}_\varepsilon^*(w_\varepsilon)]^\sim \cdot \Phi \, dx &= \int_\omega \nabla \mathcal{Q}_\varepsilon^*(w_\varepsilon) \cdot \Phi \, dx = - \int_\omega \mathcal{Q}_\varepsilon^*(w_\varepsilon) \text{div}(\Phi) \, dx \\ &= - \int_\Omega \widetilde{\mathcal{Q}}_\varepsilon^*(w_\varepsilon) \text{div}(\Phi) \, dx. \end{aligned}$$

Passing to the limit yields

$$\int_\Omega F \cdot \Phi \, dx = - \int_\Omega w \text{div}(\Phi) \, dx,$$

so that $F = \nabla w$ and w belongs to $W^{1,p}(\Omega)$.

Observe now that the restriction $\mathcal{T}_\varepsilon(\widetilde{\mathcal{Q}}_\varepsilon^*(w_\varepsilon))|_{\omega \times Y}$ belongs to $L^p(\omega; W^{1,p}(Y))$ and its gradient with respect to the variable y is of order ε . Hence, with convergence (3.28)(i) we get (3.27)(i) by the properties of \mathcal{T}_ε . Similarly, since w does not depend upon y , convergence (3.28)(ii) implies (3.27)(ii).

To prove (3.27)(iii), consider the sequence $\{\mathcal{Q}_\varepsilon^*(w_\varepsilon)|_\omega\}$. It is uniformly bounded in $W^{1,p}(\omega)$. By Theorem 1.4 applied in ω , there exists $\widehat{w}_\omega \in L^p(\omega; W_{per}^{1,p}(Y))$ such that

$$\mathcal{T}_{\varepsilon,\omega}^*(\nabla(\mathcal{Q}_\varepsilon^*(w_\varepsilon)|_\omega)) \rightharpoonup \nabla w + \nabla_y \widehat{w}_\omega \quad \text{weakly in } L^p(\omega \times Y). \quad (3.29)$$

Due to the definition of $\mathcal{Q}_\varepsilon^*$, for a.e. $x \in \omega$, the function $y \mapsto \widehat{w}_\omega(x, y)$ is a polynomial with degree less or equal to one with respect to each variable y_1, \dots, y_n . This function is also Y -periodic. Consequently, it does not depends on y . This implies

$$\mathcal{T}_\varepsilon([\nabla \mathcal{Q}_\varepsilon^*(w_\varepsilon)]^\sim) \rightharpoonup \nabla w \quad \text{weakly in } L_{loc}^p(\Omega; L^p(Y)).$$

Now, (3.27)(iii) follows since $[\nabla \mathcal{Q}_\varepsilon^*(w_\varepsilon)]^\sim$ is bounded in $L^p(\Omega)$, and so is $\mathcal{T}_\varepsilon([\nabla \mathcal{Q}_\varepsilon^*(w_\varepsilon)]^\sim)$ in $L^p(\Omega \times Y)$.

If w_ε belongs to $W_0^{1,p}(\Omega_\varepsilon^*; \Gamma_0 \cap \partial\Omega)$, consider an open set Ω' containing Ω and such that $\Gamma_0 = \Omega' \cap \partial\Omega$. We extend w_ε by zero in $(\Omega')_\varepsilon^* \setminus \Omega_\varepsilon^*$. This extension is still denoted w_ε . Due to convergences (3.28) in the context of Ω' and (3.15), it follows that w vanishes on Γ_0 . \square

We now prove a convergence result concerning the sequence $\{\mathcal{R}_\varepsilon^*(w_\varepsilon)\}$.

Lemma 3.11. *Let w_ε be in $W^{1,p}(\Omega_\varepsilon^*)$ and satisfy*

$$\|\nabla w_\varepsilon\|_{L^p(\Omega_\varepsilon^*)} \leq C. \quad (3.30)$$

Then, there exists \widehat{w}' in $L^p(\Omega; W_{per}^{1,p}(Y^))$ such that, up to a subsequence,*

$$\begin{aligned} \frac{1}{\varepsilon} \mathcal{T}_\varepsilon^*(\widetilde{\mathcal{R}_\varepsilon^*(w_\varepsilon)}) &\rightharpoonup \widehat{w}' \quad \text{weakly in } L^p(\Omega; W^{1,p}(Y^*)), \\ \mathcal{T}_\varepsilon([\nabla \mathcal{R}_\varepsilon^*(w_\varepsilon)]^\sim) &\rightharpoonup \nabla_y \widehat{w}' \quad \text{weakly in } L^p(\Omega \times Y^*), \\ \mathcal{T}_\varepsilon^*(\widetilde{\mathcal{R}_\varepsilon^*(w_\varepsilon)}) &\rightarrow 0 \quad \text{strongly in } L^p(\Omega; W^{1,p}(Y^*)). \end{aligned} \quad (3.31)$$

Proof. Due to Proposition 3.8, the sequence $\frac{1}{\varepsilon} \mathcal{T}_\varepsilon^*(\widetilde{\mathcal{R}_\varepsilon^*(w_\varepsilon)})$ is bounded in $L^p(\Omega; W^{1,p}(Y^*))$. So there exists \widehat{w}' in $L^p(\Omega; W^{1,p}(Y^*))$ such that convergences (3.31) hold.

To show that \widehat{w}' is actually in $L^p(\Omega; W_{per}^{1,p}(Y^*))$, let ω be any relatively compact open subset of Ω . The restriction of $\frac{1}{\varepsilon} \mathcal{R}_\varepsilon^*(\phi)$ to ω_ε^* belongs to $W^{1,p}(\omega_\varepsilon^*)$. Applying Theorem 3.1 to this restriction, we obtain that $\widehat{w}'|_{\omega \times Y^*}$ belongs to $L^p(\omega; W_{per}^{1,p}(Y^*))$. The conclusion follows by varying ω . \square

We are now in a position to state the main results of this section.

Theorem 3.12. *Suppose that w_ε in $W^{1,p}(\Omega_\varepsilon^*)$ satisfies*

$$\|w_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon^*)} \leq C.$$

Then, there exist w in $W^{1,p}(\Omega)$ and \widehat{w} in $L^p(\Omega; W_{per}^{1,p}(Y^))$ with $M_{Y^*}(\widehat{w}) \equiv 0$, such that, up to a subsequence,*

$$\begin{aligned} (i) \quad &\begin{cases} \mathcal{T}_\varepsilon^*(w_\varepsilon) \rightarrow w & \text{strongly in } L_{loc}^p(\Omega; W^{1,p}(Y^*)), \\ \mathcal{T}_\varepsilon^*(w_\varepsilon) \rightharpoonup w & \text{weakly in } L^p(\Omega; W^{1,p}(Y^*)), \end{cases} \\ (ii) \quad &\mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \widehat{w} \quad \text{weakly in } L^p(\Omega \times Y^*). \end{aligned} \quad (3.32)$$

Moreover, assuming that Ω is with Lipschitz boundary, if for every ε the function w_ε belongs to $W_0^{1,p}(\Omega_\varepsilon^; \Gamma_0 \cap \partial\Omega)$, then w belongs to $W_0^{1,p}(\Omega; \Gamma_0)$ (i.e. the trace of w vanishes on Γ_0).*

Proof. Convergences (3.32)(i) follow from (3.27)(i) and (ii) making use of (3.31). Similarly, convergence (ii) follows from (3.27)(iii) and (3.31) with $\widehat{w} \doteq \widehat{w}' - M_{Y^*}(\widehat{w})$. \square

Theorem 3.13. *Let Ω be bounded and with Lipschitz boundary. Suppose that w_ε belongs to $W_0^{1,p}(\Omega_\varepsilon^*; \partial\Omega \cap \partial\Omega_\varepsilon^*)$ and satisfies*

$$\|\nabla w_\varepsilon\|_{L^p(\Omega_\varepsilon^*)} \leq C.$$

Then, there exist w in $W_0^{1,p}(\Omega)$ and \widehat{w} in $L^p(\Omega; W_{per}^{1,p}(Y^))$ with $M_{Y^*}(\widehat{w}) \equiv 0$, such that, up to a subsequence,*

$$\begin{aligned} (i) \quad & \mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \widehat{w} \quad \text{weakly in } L^p(\Omega \times Y^*), \\ (ii) \quad & \mathcal{T}_\varepsilon^*(w_\varepsilon) \rightarrow w \quad \text{strongly in } L^p(\Omega; W^{1,p}(Y^*)), \\ (iii) \quad & \|w_\varepsilon - w\|_{L^p(\Omega_\varepsilon^*)} \rightarrow 0. \end{aligned} \tag{3.33}$$

Proof. By Theorem 3.9, it follows that estimate (3.26) is satisfied. Convergence (i) of (3.33) follows by Theorem 3.12. In order to obtain (ii) and (iii), without changing notations, we extend w_ε by zero to the whole of $(\mathbb{R}^n)_\varepsilon^*$ (see (2.5)). Using the unfolding operator $\mathcal{T}_{\varepsilon, \mathbb{R}^n}^*$ associated with \mathbb{R}^n , the first convergence result of Theorem 3.12 implies that $\mathcal{T}_{\varepsilon, \mathbb{R}^n}^*(w_\varepsilon)$ converges strongly in $L_{loc}^p(\mathbb{R}^n; W^{1,p}(Y^*))$. Note that $\mathcal{T}_\varepsilon^*(w_\varepsilon) = \mathcal{T}_{\varepsilon, \mathbb{R}^n}^*(w_\varepsilon) 1_{\widehat{\Omega}_\varepsilon} |_{\Omega \times Y^*}$. This implies the (ii) (3.33). As for (iii), it follows by applying Proposition 2.15 (in the setting of \mathbb{R}^n). \square

We complete this subsection with a convergence result concerning the unfolding of the difference between the general term of a sequence and its local average.

Proposition 3.14. *Suppose that w_ε in $W^{1,p}(\Omega_\varepsilon^*)$ satisfies (3.26). Assume moreover that there exist w in $W^{1,p}(\Omega)$ and \widehat{w} in $L^p(\Omega; W_{per}^{1,p}(Y^*))$ such that,*

$$\begin{aligned} \mathcal{T}_\varepsilon^*(w_\varepsilon) &\rightharpoonup w \quad \text{weakly in } L^p(\Omega; W^{1,p}(Y^*)), \\ \mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) &\rightharpoonup \nabla w + \nabla_y \widehat{w} \quad \text{weakly in } L^p(\Omega \times Y^*), \end{aligned}$$

with $\mathcal{M}_{Y^}(\widehat{w})(x) = 0$ a.e. $x \in \Omega$.*

Set $z_\varepsilon \doteq \frac{1}{\varepsilon}(w_\varepsilon - \mathcal{M}_{Y^}(\mathcal{T}_\varepsilon^*(w_\varepsilon)1_{\widehat{\Omega}_\varepsilon^*})$. Then, $\widetilde{z}_\varepsilon$ converges weakly to 0 in $L^p(\Omega)$ and*

$$Z_\varepsilon \doteq \mathcal{T}_\varepsilon^*(z_\varepsilon) = \frac{1}{\varepsilon} \left(\mathcal{T}_\varepsilon^*(w_\varepsilon) - \mathcal{M}_{Y^*}(\mathcal{T}_\varepsilon^*(w_\varepsilon)) \right) \rightharpoonup y_{\mathcal{M}} \cdot \nabla w + \widehat{w} \quad \text{weakly in } L^p(\Omega; W^{1,p}(Y^*)),$$

where $y_{\mathcal{M}} = y - \mathcal{M}_{Y^}(y)$.*

Proof. Since $\mathcal{M}_{Y^*}(\mathcal{T}_\varepsilon^*(w_\varepsilon))$ does not depend on y , $\mathcal{M}_{Y^*}(Z_\varepsilon) = 0$.

On the other hand, $\nabla_y(Z_\varepsilon) = \frac{1}{\varepsilon}\nabla_y\mathcal{T}_\varepsilon^*(w_\varepsilon) = \mathcal{T}_\varepsilon^*(\nabla w_\varepsilon)$ converges weakly to $\nabla w + \nabla_y\widehat{w}$ in $L^p(\Omega \times Y^*)$. By the Poincaré-Wirtinger inequality in $W^{1,p}(Y^*)$, it follows that the sequence $\{Z_\varepsilon\}$ is bounded in $L^p(\Omega; W^{1,p}(Y^*))$. By the connectedness of Y^* , there is a unique element in this space with zero average on Y^* whose gradient with respect to y is $\nabla w + \nabla_y\widehat{w}$, namely $y_{\mathcal{M}} \cdot \nabla w + \widehat{w}$. Consequently, $\{Z_\varepsilon\}$ converges weakly in $L^p(\Omega; W^{1,p}(Y^*))$ to $y_{\mathcal{M}} \cdot \nabla w + \widehat{w}$. By Proposition 2.8 (ii), this implies that $\widetilde{z}_\varepsilon$ converges weakly to 0 in $L^p(\Omega)$. \square

4 The boundary unfolding operator

In this section, we suppose that p is in $(1, +\infty)$, that ∂S is Lipschitz and has a finite number of connected components. The boundary of the set of holes in Ω is $\partial S_\varepsilon \cap \Omega$ and we denote by $\widehat{\partial S}_\varepsilon$ those that are included in $\widehat{\Omega}_\varepsilon$.

For a well-defined trace operator to exist from $W^{1,p}(Y^*)$ to $W^{1-1/p,p}(\partial S)$, we assume that each component of ∂S has a Lipschitz boundary. Then, a well-defined trace operator exists from $W^{1,p}(\widehat{\Omega}_\varepsilon^*)$ to $W^{1-1/p,p}(\widehat{\partial S}_\varepsilon)$.

The aim here is to give a meaning to the unfolding operator for such traces, and to obtain estimates and convergences results for sequences of functions in $W^{1,p}$ -type spaces. To do so, we extend the notion of boundary unfolding operator which was introduced in a slightly different form in [13] and [14].

Definition 4.1. For any function φ Lebesgue-measurable on $\partial\widehat{\Omega}_\varepsilon^* \cap \partial S^\varepsilon$, the boundary unfolding operator $\mathcal{T}_\varepsilon^b$ is defined by

$$\mathcal{T}_\varepsilon^b(\varphi)(x, y) = \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{a.e. for } (x, y) \in \widehat{\Omega}_\varepsilon \times \partial S, \\ 0 & \text{a.e. for } (x, y) \in \Lambda_\varepsilon \times \partial S. \end{cases} \quad (4.1)$$

Remark 4.2. If $\varphi \in W^{1,p}(\Omega_\varepsilon^*)$, $\mathcal{T}_\varepsilon^b(\varphi)$ is just the trace on ∂S of $\mathcal{T}_\varepsilon^*(\varphi)$. In particular, by the standard trace theorem in Y^* , there is a constant C such that

$$\|\mathcal{T}_\varepsilon^b(\varphi)\|_{L^p(\Omega \times \partial S)} \leq C(\|\mathcal{T}_\varepsilon^*\varphi\|_{L^p(\Omega \times Y^*)} + \|\nabla_y \mathcal{T}_\varepsilon^*\varphi\|_{L^p(\Omega \times Y^*)}). \quad (4.2)$$

From the properties of $\mathcal{T}_\varepsilon^*$, it follows that

$$\|\mathcal{T}_\varepsilon^b(\varphi)\|_{L^p(\Omega \times \partial S)} \leq C(\|\varphi\|_{L^p(\Omega_\varepsilon^*)} + \varepsilon\|\nabla\varphi\|_{L^p(\Omega_\varepsilon^*)}). \quad (4.3)$$

The operator $\mathcal{T}_\varepsilon^b$ has similar properties as the boundary unfolding operators of [13] and [14]. In particular, the integration formula, which reads

$$\int_{\widehat{\partial S}_\varepsilon} \varphi(x) d\sigma(x) = \frac{1}{\varepsilon|Y|} \int_{\Omega \times \partial S} \mathcal{T}_\varepsilon^b(\varphi)(x, y) dx d\sigma(y), \quad (4.4)$$

transforms an integral on the rapidly oscillating set $\widehat{\partial S}_\varepsilon$ into an integral on a fixed set $\Omega \times \partial S$. The integration formula implies

$$\|\mathcal{T}_\varepsilon^b(\varphi)\|_{L^p(\Omega \times \partial S)} = \varepsilon^{1/p} |Y|^{1/p} \|\varphi\|_{L^p(\widehat{\partial S}_\varepsilon)}. \quad (4.5)$$

The presence of the power of ε in (4.4) requires a normalization for boundary terms which differs from that in the bulk. This induces some interesting effects for the convergence of such boundary integrals (see Propositions 4.6 and 4.10).

Proposition 4.3. *Suppose that v belongs to $W^{1,p}(\Omega_\varepsilon^*)$ and that g is in $L^{p'}(\widehat{\partial S}_\varepsilon)$. Then*

$$\begin{aligned} \left| \int_{\widehat{\partial S}_\varepsilon} g v \, d\sigma(x) \right| &\leq C \left(\|\mathcal{T}_\varepsilon^b(g)\|_{L^{p'}(\Omega \times \partial S)} \|\nabla v\|_{L^p(\Omega_\varepsilon^*)} + \frac{1}{\varepsilon} \|\mathcal{M}_{\partial S}(\mathcal{T}_\varepsilon^b(g))\|_{L^{p'}(\Omega)} \|v\|_{L^p(\Omega_\varepsilon^*)} \right), \\ \left| \int_{\widehat{\partial S}_\varepsilon} g v \, d\sigma(x) \right| &\leq \frac{C}{\varepsilon^{1/p}} \|g\|_{L^{p'}(\widehat{\partial S}_\varepsilon)} \left(\|v\|_{L^p(\Omega_\varepsilon^*)} + \varepsilon \|\nabla v\|_{L^p(\Omega_\varepsilon^*)} \right). \end{aligned}$$

Proof. Applying (4.4) to the product $g v$ gives

$$\int_{\widehat{\partial S}_\varepsilon} g v \, d\sigma(x) = \frac{1}{\varepsilon |Y|} \int_{\Omega \times \partial S} \mathcal{T}_\varepsilon^b(g)(x, y) \mathcal{T}_\varepsilon^*(v)(x, y) \, dx \, d\sigma(y). \quad (4.6)$$

This can be written as

$$\begin{aligned} \int_{\widehat{\partial S}_\varepsilon} g v \, d\sigma(x) &= \frac{1}{\varepsilon |Y|} \int_{\Omega \times \partial S} \mathcal{T}_\varepsilon^b(g)(x, y) (\mathcal{T}_\varepsilon^*(v) - \mathcal{M}_{Y^*}(\mathcal{T}_\varepsilon^*(v)))(x, y) \, dx \, d\sigma(y) \\ &\quad + \frac{|\partial S|}{\varepsilon |Y|} \int_{\Omega} \mathcal{M}_{\partial S}(\mathcal{T}_\varepsilon^b(g))(x) \mathcal{M}_{Y^*}(\mathcal{T}_\varepsilon^*(v))(x) \, dx. \end{aligned} \quad (4.7)$$

By the Hölder inequality,

$$\begin{aligned} \left| \int_{\widehat{\partial S}_\varepsilon} g v \, d\sigma(x) \right| &\leq \frac{C}{\varepsilon} \|\mathcal{T}_\varepsilon^b(g)\|_{L^{p'}(\Omega \times \partial S)} \|\mathcal{T}_\varepsilon^*(v) - \mathcal{M}_{Y^*}(\mathcal{T}_\varepsilon^*(v))\|_{L^p(\Omega \times \partial S)} \\ &\quad + \frac{C}{\varepsilon} \|\mathcal{M}_{\partial S}(\mathcal{T}_\varepsilon^b(g))\|_{L^{p'}(\Omega)} \|\mathcal{M}_{Y^*}(\mathcal{T}_\varepsilon^*(v))\|_{L^p(\Omega)}. \end{aligned}$$

On the one hand, due to the Poincaré-Wirtinger inequality in Y^* ,

$$\|\mathcal{T}_\varepsilon^*(v) - \mathcal{M}_{Y^*}(\mathcal{T}_\varepsilon^*(v))\|_{L^p(\Omega \times \partial S)} \leq C \|\nabla_y \mathcal{T}_\varepsilon^*(v)\|_{L^p(\Omega \times Y^*)} \leq C \varepsilon \|\nabla v\|_{L^p(\Omega_\varepsilon^*)}.$$

On the other hand, we have

$$\|\mathcal{M}_{Y^*}(\mathcal{T}_\varepsilon^*(v))\|_{L^p(\Omega)} \leq C \|\mathcal{T}_\varepsilon^*(v)\|_{L^p(\Omega)} \leq C \|v\|_{L^p(\Omega_\varepsilon^*)}.$$

The first inequality follows.

Since $\|\mathcal{M}_{\partial S}(\mathcal{T}_\varepsilon^b(g))\|_{L^{p'}(\Omega)} \leq C \|\mathcal{T}_\varepsilon^b(g)\|_{L^{p'}(\widehat{\Omega}_\varepsilon \times \partial S)}$, the second inequality follows from the first one and from (4.5). \square

A simple consequence of formula (4.6) is the following convergence result:

Proposition 4.4. *Suppose w_ε is in $W^{1,p}(\Omega_\varepsilon^*)$, g_ε is in $L^{p'}(\widehat{\partial S}_\varepsilon)$ and*

$$\begin{aligned} (i) \quad \mathcal{T}_\varepsilon^*(w_\varepsilon) &\rightharpoonup w \quad \text{weakly in } L^p(\Omega; W^{1,p}(Y^*)), \\ (ii) \quad \mathcal{T}_\varepsilon^b(g_\varepsilon) &\rightarrow g \quad \text{strongly in } L^{p'}(\Omega \times \partial S). \end{aligned} \tag{4.8}$$

Then

$$\varepsilon \int_{\widehat{\partial S}_\varepsilon} g_\varepsilon w_\varepsilon d\sigma(x) \rightarrow \frac{1}{|Y|} \int_{\Omega \times \partial S} g(x, y) w(x, y) dx d\sigma(y). \tag{4.9}$$

Remark 4.5.

- (i) *It is obvious that strong and weak convergences in Proposition 4.4 can be interchanged.*
- (ii) *For any g in $L^{p'}(\Omega \times \partial S)$, there is always a sequence $\{g_\varepsilon\}$ satisfying (4.8)(ii) (see also Remark 4.8 below for a more general statement).*
- (iii) *In Proposition 4.4, if w is independent of y and $\mathcal{M}_{\partial S}(g) = 0$, then*

$$\varepsilon \int_{\widehat{\partial S}_\varepsilon} g_\varepsilon w_\varepsilon d\sigma(x) \rightarrow 0.$$

The last result ((iii) of Remark 4.5), for w independent of y and $\mathcal{M}_{\partial S}(g) = 0$, was already observed in [13]. One can obtain the limit of the integral $\int_{\widehat{\partial S}_\varepsilon} g_\varepsilon w_\varepsilon d\sigma(x)$ itself under some additional assumptions. This is given in the next result.

Proposition 4.6. *Let w_ε be in $W^{1,p}(\Omega_\varepsilon^*)$. Suppose there exist w in $W^{1,p}(\Omega)$ and \widehat{w} in $L^p(\Omega; W_{per}^{1,p}(Y^*))$ such that,*

$$\begin{aligned} \mathcal{T}_\varepsilon^*(w_\varepsilon) &\rightharpoonup w \quad \text{weakly in } L^p(\Omega; W^{1,p}(Y^*)), \\ \mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) &\rightharpoonup \nabla w + \nabla_y \widehat{w} \quad \text{weakly in } L^p(\Omega \times Y^*), \end{aligned} \tag{4.10}$$

with $\mathcal{M}_{Y^}(\widehat{w})(x) = 0$ for a.e. $x \in \Omega$. Suppose also that the following two convergences hold for g_ε in $L^{p'}(\widehat{\partial S}_\varepsilon)$:*

$$\begin{aligned} \mathcal{T}_\varepsilon^b(g_\varepsilon) &\rightarrow g \quad \text{strongly in } L^{p'}(\Omega \times \partial S), \\ \frac{1}{\varepsilon} \mathcal{M}_{\partial S}(\mathcal{T}_\varepsilon^b(g_\varepsilon)) &\rightarrow G \quad \text{strongly in } L^{p'}(\Omega). \end{aligned} \tag{4.11}$$

Then,

$$\begin{aligned} \int_{\widehat{\partial S}_\varepsilon} g_\varepsilon w_\varepsilon d\sigma(x) &\rightarrow \frac{|\partial S|}{|Y|} \int_{\Omega} G w dx + \frac{|\partial S|}{|Y|} \int_{\Omega} \mathcal{M}_{\partial S}(y_{\mathcal{M}} g) \cdot \nabla w dx \\ &\quad + \frac{1}{|Y|} \int_{\Omega \times \partial S} g \widehat{w} dx d\sigma(y), \end{aligned} \quad (4.12)$$

where $y_{\mathcal{M}} = y - \mathcal{M}_{Y^*}(y)$.

Proof. Formula (4.7) gives

$$\begin{aligned} \int_{\widehat{\partial S}_\varepsilon} g_\varepsilon w_\varepsilon d\sigma(x) &= \frac{|\partial S|}{|Y|} \int_{\Omega} \frac{1}{\varepsilon} \mathcal{M}_{\partial S}(\mathcal{T}_\varepsilon^b(g_\varepsilon))(x) \mathcal{M}_{Y^*}(\mathcal{T}_\varepsilon^*(w_\varepsilon))(x) dx \\ &\quad + \frac{1}{|Y|} \int_{\Omega \times \partial S} \frac{1}{\varepsilon} \left(\mathcal{T}_\varepsilon^*(w_\varepsilon) - \mathcal{M}_{Y^*}(\mathcal{T}_\varepsilon^*(w_\varepsilon)) \right)(x, y) \mathcal{T}_\varepsilon^b(g_\varepsilon)(x, y) dx d\sigma(y). \end{aligned}$$

Then convergence (4.12) is obtained from the fact that $\mathcal{M}_{Y^*}(\mathcal{T}_\varepsilon^*(w_\varepsilon))$ converges weakly in $L^p(\Omega)$ to w (since $\mathcal{T}_\varepsilon^*(w_\varepsilon)$ itself converges to w which is independent of y) for the first term and from Proposition 3.14 for the second term (which itself gives two terms in the limit). \square

Remark 4.7. *Actually, Proposition 4.6 can be stated with the same conclusion with four different sets of hypotheses by exchanging weak and strong convergences in (4.10) and (4.11). We will use the same reference for these alternate versions. An interesting case is when convergences 4.10 are both strong. Then convergence (4.12) holds assuming only weak convergences in (4.11).*

Remark 4.8. *Note that if g_ε in $L^{p'}(\widehat{\partial S}_\varepsilon)$ is such that*

$$\begin{aligned} \mathcal{T}_\varepsilon^b(g_\varepsilon) &\rightharpoonup g \quad \text{weakly in } L^{p'}(\Omega \times \partial S), \\ \frac{1}{\varepsilon} \mathcal{M}_{\partial S}(\mathcal{T}_\varepsilon^b(g_\varepsilon)) &\rightharpoonup G \quad \text{weakly in } L^{p'}(\Omega), \end{aligned}$$

it follows that

$$\mathcal{M}_{\partial S}(g) = 0 \quad \text{a.e. for } x \in \Omega.$$

Conversely, for any g in $L^{p'}(\Omega \times \partial S)$ with $\mathcal{M}_{\partial S}(g)(x) = 0$ for a.e. $x \in \Omega$, and G in $L^{p'}(\Omega)$, there always exists a sequence g_ε satisfying (4.11). An example of such a sequence is given by

$$g_\varepsilon = \mathcal{U}_\varepsilon^b(g) + \varepsilon \mathcal{M}_{Y^*}(\mathcal{T}_\varepsilon(G))|_{\widehat{\partial S}_\varepsilon}.$$

We now extend the result of Propositions 4.3 and 4.6 to the case where w_ε belongs to $W_0^{1,p}(\Omega_\varepsilon^*; \partial\Omega \cap \partial\Omega_\varepsilon^*)$ and g_ε to $L_{loc}^{p'}(\partial S_\varepsilon)$. Recall that $\mathcal{V}_1(\Omega)$ denotes the 1-neighborhood of Ω .

Proposition 4.9. *Assume that Ω is bounded with Lipschitz boundary. Let w belong to $W_0^{1,p}(\Omega_\varepsilon^*; \partial\Omega \cap \partial\Omega_\varepsilon^*)$ and g to $L_{loc}^{p'}(\partial S_\varepsilon)$. Then, for ε small enough*

$$\begin{aligned} \left| \int_{\partial S_\varepsilon \cap \Omega} g w d\sigma(x) \right| &\leq C \|\mathcal{T}_\varepsilon^b(g)\|_{L^{p'}(\mathcal{V}_1(\Omega) \times \partial S)} \|\nabla w\|_{L^p(\Omega_\varepsilon^*)} \\ &\quad + \frac{C}{\varepsilon} \|\mathcal{M}_{\partial S}(\mathcal{T}_\varepsilon^b(g))\|_{L^{p'}(\mathcal{V}_1(\Omega))} \|w\|_{L^p(\Omega_\varepsilon^*)}, \\ \left| \int_{\partial S_\varepsilon \cap \Omega} g w d\sigma(x) \right| &\leq \frac{C}{\varepsilon^{1/p}} \|g\|_{L^{p'}(\partial S_\varepsilon \cap \mathcal{V}_1(\Omega))} \left(\|w\|_{L^p(\Omega_\varepsilon^*)} + \varepsilon \|\nabla w\|_{L^p(\Omega_\varepsilon^*)} \right). \end{aligned}$$

Proposition 4.10. *Let Ω be a bounded domain with Lipschitz boundary and w_ε be in $W_0^{1,p}(\Omega_\varepsilon^*; \partial\Omega \cap \partial\Omega_\varepsilon^*)$, satisfying $\|\nabla w_\varepsilon\|_{L^p(\Omega_\varepsilon^*)} \leq C$. Suppose that there exist w in $W_0^{1,p}(\Omega)$ and \widehat{w} in $L^p(\Omega; W_{per}^{1,p}(Y^*))$ such that,*

$$\begin{aligned} \mathcal{T}_\varepsilon^*(w_\varepsilon) &\rightharpoonup w \quad \text{weakly in } L^p(\Omega; W^{1,p}(Y^*)), \\ \mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) &\rightharpoonup \nabla w + \nabla_y \widehat{w} \quad \text{weakly in } L^p(\Omega \times Y^*), \end{aligned} \tag{4.13}$$

with $\mathcal{M}_{Y^*}(\widehat{w})(x) = 0$ a.e. $x \in \Omega$.

Let $g_\varepsilon \in L_{loc}^{p'}(\partial S_\varepsilon)$ and suppose furthermore that

$$\begin{aligned} \mathcal{T}_\varepsilon^b(g_\varepsilon) &\rightarrow g \quad \text{strongly in } L_{loc}^{p'}(\mathbb{R}^n \times \partial S), \\ \frac{1}{\varepsilon} \mathcal{M}_{\partial S}(\mathcal{T}_\varepsilon^b(g_\varepsilon)) &\rightarrow G \quad \text{strongly in } L_{loc}^{p'}(\mathbb{R}^n) \end{aligned} \tag{4.14}$$

(where $\mathcal{T}_\varepsilon^b$ acts in $(\mathbb{R}^n)_\varepsilon^*$; see (2.5)). Then we have

$$\begin{aligned} \int_{\partial S_\varepsilon \cap \Omega} g_\varepsilon w_\varepsilon d\sigma(x) &\rightarrow \frac{|\partial S|}{|Y|} \int_{\Omega} \mathcal{M}_{\partial S}(y_{\mathcal{M}} g) \cdot \nabla w dx + \frac{|\partial S|}{|Y|} \int_{\Omega} G w dx \\ &\quad + \frac{1}{|Y|} \int_{\Omega \times \partial S} \widehat{w} g dx d\sigma(y). \end{aligned} \tag{4.15}$$

Proof of Propositions 4.9 and 4.10. Extend w and w_ε by 0 to the whole $(\mathbb{R}^n)_\varepsilon^*$ to get

$$\int_{\partial S_\varepsilon \cap \Omega} g_\varepsilon w_\varepsilon d\sigma(x) = \int_{(\widehat{\partial S_\varepsilon})_1} g_\varepsilon w_\varepsilon d\sigma(x), \tag{4.16}$$

where $(\widehat{\partial S_\varepsilon})_1$ is the $\widehat{\partial S_\varepsilon}$ associated with $\mathcal{V}_1(\Omega)$ instead of Ω (with a similar relation for w). The results then follow from Propositions 4.3 and 4.6 applied in the domain $\mathcal{V}_1(\Omega)$. \square

Remark 4.11. In the literature, there are two standard examples of periodic functions g_ε , deriving from a function \mathbf{g} in $L^p(\partial S)$.

For Hypothesis (4.14), g_ε is defined as

1. $g_\varepsilon(x) = \varepsilon \mathbf{g}(\{x/\varepsilon\}_Y)$ if $\mathcal{M}_{\partial S}(\mathbf{g}) \neq 0$,
2. $g_\varepsilon(x) = \mathbf{g}(\{x/\varepsilon\}_Y)$ if $\mathcal{M}_{\partial S}(\mathbf{g}) = 0$.

For Hypothesis (4.11), since the functions g_ε have to vanish outside of $\widehat{\partial S_\varepsilon}$, the formulas are

3. $g_\varepsilon(x) = \varepsilon \mathbf{g}(\{x/\varepsilon\}_Y) 1_{\widehat{S_\varepsilon}}$ if $\mathcal{M}_{\partial S}(\mathbf{g}) \neq 0$,
4. $g_\varepsilon(x) = \mathbf{g}(\{x/\varepsilon\}_Y) 1_{\widehat{S_\varepsilon}}$ if $\mathcal{M}_{\partial S}(\mathbf{g}) = 0$.

At the limit, in cases 1 and 3, $g = 0$, $G = \mathcal{M}_{\partial S}(\mathbf{g})$, while in cases 2 and 4, $g = \mathbf{g}$ and $G = 0$. Note that there is no way to have both g and G non zero by simply using the periodic approach.

Remark 4.12. The setting of [13] is restricted to the case 1 above. In particular, if $\mathcal{M}_{\partial S}(g) = 0$, Proposition 4.10 holds with $G = 0$. Moreover, by Proposition 4.10 with $w_\varepsilon \equiv w$, one immediately has

$$\int_{\partial S_\varepsilon \cap \Omega} g_\varepsilon w d\sigma(x) \rightarrow 0,$$

which should be compared with $\varepsilon \int_{\widehat{\partial S_\varepsilon}} g_\varepsilon w d\sigma(x) \rightarrow 0$ (see (iii) of Remark 4.5).

5 Application: homogenization in periodically perforated domains

We present two generalizations of classical homogenization problems in bounded domains with holes. In the both cases, the boundary condition on the holes is of non homogeneous Neumann type. In the first problem, the condition on the outer boundary is homogeneous Dirichlet (Dirichlet-Neumann problem) while in the second problem it is homogeneous Neumann.

Throughout this section, we assume that Ω is bounded with Lipschitz boundary and that Hypothesis **(H₂)** holds.

Let f be in $L^2(\Omega)^1$ and $A^\varepsilon(x) = (a_{ij}^\varepsilon(x))_{1 \leq i, j \leq n}$ be a matrix field in the set $M(\alpha, \beta, \Omega)$, according to the standard definition below.

¹One can as easily consider a sequence $\{f_\varepsilon\}$ which converges weakly to f in $L^2(\Omega)$. However, one cannot choose a fixed f in $H^{-1}(\Omega)$, simply because it cannot be restricted Ω_ε^* in a meaningful way.

Definition 5.1. Let $\alpha, \beta \in \mathbb{R}$, such that $0 < \alpha < \beta$. $M(\alpha, \beta, \mathcal{O})$ denotes the set of the $n \times n$ matrices $B = B(x)$, $B = (b_{ij})_{1 \leq i, j \leq n} \in (L^\infty(\mathcal{O}))^{n \times n}$ such that for any $\lambda \in \mathbb{R}^n$ and a.e. on \mathcal{O} ,

$$(B(x)\lambda, \lambda) \geq \alpha|\lambda|^2, \quad |A(x)\lambda| \leq \beta|\lambda|.$$

The Dirichlet-Neumann problem is

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega_\varepsilon^*, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon^* \cap \partial\Omega, \\ A^\varepsilon u_\varepsilon \cdot n_\varepsilon = g_\varepsilon & \text{on } \partial S_\varepsilon \cap \Omega, \end{cases} \quad (5.1)$$

where g_ε is given in $L^2(\partial S_\varepsilon \cap \Omega)$.

The Neumann problem is

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla u_\varepsilon) + b_\varepsilon u_\varepsilon = f & \text{in } \Omega_\varepsilon^*, \\ A^\varepsilon u_\varepsilon \cdot n_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon^* \setminus \widehat{\partial S_\varepsilon}, \\ A^\varepsilon u_\varepsilon \cdot n_\varepsilon = g_\varepsilon & \text{on } \widehat{\partial S_\varepsilon}, \end{cases} \quad (5.2)$$

where the function b_ε is measurable, positive a.e. in Ω , essentially bounded as well as its inverse, and g_ε is given in $L^2(\partial S_\varepsilon)$ (see notation (2.4)).

Remark 5.2. As far as we know, there is no homogenization result for Problem 5.2 if g_ε is defined on the whole of ∂S_ε and does not vanish outside of $\widehat{\partial S_\varepsilon}$. This is mainly due to the lack of uniform bounds for solutions. In some papers, the holes in the zone $\partial S_\varepsilon \setminus \widehat{\partial S_\varepsilon}$ (or in a similar boundary layer) are completely suppressed, in which case the unfolding approach works as easily (since all unfolded functions always vanish in this layer).

The variational formulation of (5.1) is

$$\begin{cases} \text{Find } u_\varepsilon \in H_0^1(\Omega_\varepsilon^*; \partial\Omega \cap \partial\Omega_\varepsilon^*) \text{ such that} \\ \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \cdot \nabla v \, dx = \int_{\Omega_\varepsilon^*} f v \, dx + \int_{\partial S_\varepsilon \cap \Omega} g_\varepsilon v \, d\sigma(x), \\ \forall v \in H_0^1(\Omega_\varepsilon^*; \partial\Omega \cap \partial\Omega_\varepsilon^*). \end{cases} \quad (5.3)$$

The variational formulation of (5.2) is

$$\begin{cases} \text{Find } u_\varepsilon \in H^1(\Omega_\varepsilon^*) \text{ such that} \\ \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \cdot \nabla v \, dx + \int_{\Omega_\varepsilon^*} b_\varepsilon u_\varepsilon v \, dx = \int_{\Omega_\varepsilon^*} f v \, dx + \int_{\widehat{\partial S_\varepsilon}} g_\varepsilon v \, d\sigma(x), \\ \forall v \in H^1(\Omega_\varepsilon^*). \end{cases} \quad (5.4)$$

A major difficulty when considering (5.3) and (5.4), is the strong dependence upon ε of the spaces $H_0^1(\Omega_\varepsilon^*; \partial\Omega \cap \partial\Omega_\varepsilon^*)$ and $H^1(\Omega_\varepsilon^*)$.

What kind of convergence can be expected for the sequence $\{u_\varepsilon\}$?

One approach is to use uniformly bounded extension operators P_ε from $H^1(\Omega_\varepsilon^*)$ to $H^1(\Omega)$ (respectively from $H_0^1(\Omega_\varepsilon^*; \partial\Omega \cap \partial\Omega_\varepsilon^*)$ to $H_0^1(\Omega)$). The weak convergence of $\{P_\varepsilon(u_\varepsilon)\}$ in the corresponding fixed space can then be proved. This is the case for sufficiently smooth holes not intersecting the boundary of Ω . Such extension operators are constructed on the unit cell under restrictive conditions on the normalized hole S (for this approach, we refer the reader to [4], [15], [16] and the references therein, [23], [5], [7], see also [18]). The choice of the cell Y can be critical. For example, Figure 4 shows two possible choices of unit cell, which differ only by the position of the cell with respect to the origin of \mathbb{R}^n (the hole S is the same in both cases). The problems are therefore identical. But for the one on the left, provided the hole is with Lipschitz boundary, one can construct such an extension operator. For the choice on the right no such extension operator can be constructed!

Without such extension operators, even if $\|u_\varepsilon\|_{H^1(\Omega_\varepsilon^*)}$ is uniformly bounded, one cannot speak about “convergence” of u_ε . For homogeneous Neumann condition on the boundary of the holes, a first attempt in this direction was made in [3] where the obtained convergence was the following: $\|u_\varepsilon - u\|_{L^2(\Omega_\varepsilon^*)} \rightarrow 0$ for the Dirichlet-Neumann problem (resp. $\forall \omega \subset\subset \Omega$, $\|u_\varepsilon - u\|_{L^2(\omega \cap \Omega_\varepsilon^*)} \rightarrow 0$).

In contrast, in the unfolding method, the sequences $\{\mathcal{T}_\varepsilon^*(u_\varepsilon)\}$ and $\{\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon)\}$ are bounded in the fixed spaces $L^2(\Omega; H^1(Y^*))$ and $L^2(\Omega \times Y^*)$, thereby allowing the use of standard convergences. These convergences, in turn give information on the original sequences as well as corrector results (see (5.9) and (5.44)).

In the above variational formulations, the presence of integrals on the perforation boundaries requires the existence of traces on ∂S and is another difficulty because these boundaries vary wildly with ε . This is overcome by the use of the boundary unfolding operator which rewrites them as integrals on the fixed set $\Omega \times \partial S$. In the particular case of periodic boundary data of the form $g_\varepsilon(x) = g(\{x/\varepsilon\}_Y)$ for some g in $L^{p'}(\partial S)$, this procedure was used in [13]-[14] (see also Remark 4.11).

In each problem, under strong convergence conditions on the data, we obtain a corrector result which is new in this context.

5.1 Homogenization of the Dirichlet-Neumann problem

When studying the asymptotic behavior of (5.1), the first point is to obtain a uniform bound for u_ε solution of (5.3). To do so, we first choose an extension of g_ε to $\partial S_\varepsilon \cap \mathcal{V}_1(\Omega)$ (still using the notation g_ε for this extension).² Then we choose u_ε as a test function in (5.3). From the resulting formula, making use of Proposition 4.9 and Theorem 3.9 (for

²Actually, any extension in $L^2(\partial S_\varepsilon \cap \mathcal{V}_1(\Omega))$ will do. We usually choose either the periodic extension (if it is possible), or the extension by 0. Recall that $\mathcal{V}_1(\Omega)$ is the 1-neighborhood of Ω .

$p = 2$), one can establish the uniform bound

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon^*)} \leq C(\|f\|_{L^2(\Omega_\varepsilon^*)} + \varepsilon^{1/2}\|g_\varepsilon\|_{L^2(\partial S_\varepsilon \cap \mathcal{V}_1(\Omega))} + \frac{1}{\varepsilon}\|\mathcal{M}_{\partial S}(\mathcal{T}_\varepsilon^b(g_\varepsilon))\|_{L^2(\mathcal{V}_1(\Omega))}). \quad (5.5)$$

In view of this estimate, the natural condition on the function g_ε is that

$$\varepsilon^{1/2}\|g_\varepsilon\|_{L^2(\partial S_\varepsilon \cap \mathcal{V}_1(\Omega))} + \frac{1}{\varepsilon}\|\mathcal{M}_{\partial S}(\mathcal{T}_\varepsilon^b(g_\varepsilon))\|_{L^2(\mathcal{V}_1(\Omega))} \quad \text{is uniformly bounded,} \quad (5.6)$$

which gives a uniform bound for $\|u_\varepsilon\|_{H^1(\Omega_\varepsilon^*)}$. We can now state the homogenization result.

Theorem 5.3. *Let u_ε be the solution of problem (5.1). Suppose that*

$$\mathcal{T}_\varepsilon^*(A^\varepsilon) \rightarrow A \quad \text{a.e. in } \Omega \times Y^* \quad (\text{or more generally, in measure in } \Omega \times Y^*), \quad (5.7)$$

for some matrix $A = A(x, y)$ such that

$$A = (a_{ij})_{1 \leq i, j \leq n} \in M(\alpha, \beta, \Omega \times Y^*).$$

Suppose furthermore that g_ε satisfies (5.6) and that there exist g in $L^2(\Omega \times \partial S)$ and G in $L^2(\Omega)$ satisfying

$$\begin{aligned} \mathcal{T}_\varepsilon^b(g_\varepsilon) &\rightharpoonup g \quad \text{weakly in } L^2(\Omega \times \partial S), \\ \frac{1}{\varepsilon}\mathcal{M}_{\partial S}(\mathcal{T}_\varepsilon^b(g_\varepsilon)) &\rightharpoonup G \quad \text{weakly in } L^2(\Omega). \end{aligned} \quad (5.8)$$

Then, there exist $u_0 \in H_0^1(\Omega)$ and $\widehat{u}_0 \in L^2(\Omega; H_{per}^1(Y^*))$ such that

$$\begin{aligned} (i) \quad &\|u_\varepsilon - u_0\|_{L^2(\Omega_\varepsilon^*)} \rightarrow 0, \\ (ii) \quad &\mathcal{T}_\varepsilon^*(u_\varepsilon) \rightarrow u_0 \quad \text{strongly in } L^2(\Omega; H^1(Y^*)), \\ (iii) \quad &\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \rightharpoonup \nabla u_0 + \nabla_y \widehat{u}_0 \quad \text{weakly in } L^2(\Omega \times Y^*), \end{aligned} \quad (5.9)$$

and the pair (u_0, \widehat{u}_0) is the unique solution of the problem

$$\left\{ \begin{aligned} &u_0 \in H_0^1(\Omega), \quad \widehat{u}_0 \in L^2(\Omega; H_{per}^1(Y^*)) \text{ with } \mathcal{M}_{Y^*}(\widehat{u}_0) = 0 \quad \text{for a.e. } x \in \Omega, \\ &\frac{1}{|Y|} \int_{\Omega \times Y^*} A(x, y) [\nabla u_0(x) + \nabla_y \widehat{u}_0(x, y)] [\nabla \Psi(x) + \nabla_y \Phi(x, y)] \, dx dy \\ &\quad = \frac{|Y^*|}{|Y|} \int_{\Omega} f(x) \Psi(x) \, dx + \frac{|\partial S|}{|Y|} \int_{\Omega} \mathcal{M}_{\partial S}(y_{\mathcal{M}} g)(x) \cdot \nabla \Psi(x) \, dx \\ &\quad \quad + \frac{|\partial S|}{|Y|} \int_{\Omega} G(x) \Psi(x) \, dx + \frac{1}{|Y|} \int_{\Omega \times \partial S} g(x, y) \Phi(x, y) \, dx \, d\sigma(y), \\ &\forall \Psi \in H_0^1(\Omega), \quad \forall \Phi \in L^2(\Omega; H_{per}^1(Y^*)). \end{aligned} \right. \quad (5.10)$$

Remark 5.4. *As in the case of fixed domains (see [10]), every matrix field A belonging to $M(\alpha, \beta, \Omega \times Y^*)$ can be approached (in the sense of (5.7)) by the sequence of matrices A^ε in $M(\alpha, \beta, \Omega_\varepsilon^*)$ with A^ε defined as follows:*

$$A^\varepsilon = \begin{cases} \mathcal{U}_\varepsilon(A) & \text{in } \widehat{\Omega}_\varepsilon^* \\ \alpha I_n & \text{in } \Lambda_\varepsilon^*. \end{cases}$$

Proof of Theorem 5.3. First, note that problem (5.10) has a solution which is unique by direct application of the Lax-Milgram theorem in the space $H_0^1(\Omega) \times L^2(\Omega; H_{per}^1(Y^*)/\mathbb{R})$ ($H_{per}^1(Y^*)/\mathbb{R}$ is identified with the closed subspace of $H_{per}^1(Y^*)$ consisting of all its functions with mean value 0).

As seen above, (5.5) and (5.6) imply that $\|u_\varepsilon\|_{H^1(\Omega_\varepsilon^*)}$ is uniformly bounded. Then, Theorems 3.12 and 3.13 imply convergences (5.9), at least for a subsequence. The uniqueness of the solution of the limit problem will eventually imply the convergence of the whole sequence.

Let Ψ and φ be in $\mathcal{D}(\Omega)$ and $\psi = \psi(y)$ in $H_{per}^1(Y^*)$ such that $\mathcal{M}_{Y^*}(\psi) = 0$. We choose in (5.3) the test function $v^\varepsilon = \Psi + \varepsilon\varphi\psi_\varepsilon$ where $\psi_\varepsilon(x) = \psi\left(\frac{x}{\varepsilon}\right)$. Since

$$\nabla v^\varepsilon = \nabla_x \Psi(x) + \varepsilon\psi_\varepsilon \nabla_x \varphi + \varphi(\nabla_y \psi)\left(\frac{\cdot}{\varepsilon}\right),$$

by Proposition 2.8 (i),

$$\begin{aligned} \mathcal{T}_\varepsilon^*(v^\varepsilon) &\rightarrow \Psi \quad \text{strongly in } L^2(\Omega \times Y^*), \\ \mathcal{T}_\varepsilon^*(\varphi\psi_\varepsilon) &\rightarrow \Phi \quad \text{strongly in } L^2(\Omega \times Y^*), \quad \text{with } \Phi(x, y) = \varphi(x)\psi(y), \\ \mathcal{T}_\varepsilon^*(\nabla v^\varepsilon) &\rightarrow \nabla \Psi + \nabla_y \Phi \quad \text{strongly in } L^2(\Omega \times Y^*). \end{aligned} \tag{5.11}$$

Then, for ε small enough, Proposition 2.6 (i) and (5.9) allow passing to the limit to get

$$\begin{aligned} \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v^\varepsilon \, dx &= \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(A^\varepsilon) \mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \mathcal{T}_\varepsilon^*(\nabla v^\varepsilon) \, dx \, dy \\ &\rightarrow \frac{1}{|Y|} \int_{\Omega \times Y^*} A(x, y) [\nabla u_0(x) + \nabla_y \widehat{u}_0(x, y)] [\nabla \Psi(x) + \phi(x) \nabla \psi(y)] \, dx \, dy, \end{aligned} \tag{5.12}$$

as well as

$$\int_{\Omega_\varepsilon^*} f v^\varepsilon \, dx = \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(v^\varepsilon) \, dx \, dy \longrightarrow \frac{|Y^*|}{|Y|} \int_{\Omega} f(x) \Psi(x) \, dx. \tag{5.13}$$

For ε small enough, in view of the fact that the support of v_ε remains in a fixed compact subset of Ω , the surface integral in (5.3) takes the form

$$\int_{\partial S_\varepsilon \cap \Omega} g_\varepsilon v_\varepsilon \, d\sigma(x) = \int_{\widehat{\partial S}_\varepsilon} g_\varepsilon \Psi \, d\sigma(x) + \varepsilon \int_{\widehat{\partial S}_\varepsilon} g_\varepsilon \varphi \psi_\varepsilon \, d\sigma(x).$$

To pass to the limit here, we use the results of Section 6. In view of hypotheses (5.8) and convergences (5.11), we can apply Proposition 4.6 in the form of Remark 4.7, to the first integral in the right hand side to obtain

$$\int_{\widehat{\partial S_\varepsilon}} g_\varepsilon \Psi d\sigma(x) \rightarrow \frac{|\partial S|}{|Y|} \int_{\Omega} \mathcal{M}_{\partial S}(y_{\mathcal{M}} g) \cdot \nabla \Psi dx + \frac{|\partial S|}{|Y|} \int_{\Omega} G \Psi dx. \quad (5.14)$$

For the second integral, Proposition 4.4 and (5.11) again give

$$\varepsilon \int_{\widehat{\partial S_\varepsilon}} g_\varepsilon \varphi \psi_\varepsilon d\sigma(x) \rightarrow \frac{1}{|Y|} \int_{\Omega \times \partial S} g(x) \varphi(x) \psi(y) dx d\sigma(y). \quad (5.15)$$

Collecting (5.12), (5.13), (5.14) and (5.15), and due to the density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$ and that of the tensor product $\mathcal{D}(\Omega) \otimes H_{per}^1(Y^*)$ in $L^2(\Omega; H_{per}^1(Y^*))$, we get the unfolded limit formulation (5.10). \square

We now turn to the “classical” formulation of the limit problem in terms of u_0 alone. We begin by expressing \widehat{u}_0 in terms of u_0 .

Proposition 5.5. *The function \widehat{u}_0 in Theorem 5.3 is given in terms of u_0 by*

$$\widehat{u}_0(x, y) = \sum_{i=1}^n \frac{\partial u_0}{\partial x_i}(x) \chi_i(x, y) + \chi_0(x, y), \quad (5.16)$$

where the corrector functions $\chi_j \in L^\infty(\Omega; H_{per}^1(Y^*))$ ($j = 1, \dots, n$), are, for a.e. x in Ω , the solutions of the cell problems

$$\begin{cases} - \sum_{i,k=1}^n \frac{\partial}{\partial y_i} \left(a_{ik}(x, y) \left(\frac{\partial \chi_j(x, y)}{\partial y_k} - \delta_{jk} \right) \right) = 0 & \text{in } Y^*, \\ \sum_{i,k=1}^n a_{ik}(x, y) \left(\frac{\partial \chi_j(x, y)}{\partial y_k} - \delta_{jk} \right) n_i = 0 & \text{on } \partial S, \\ \mathcal{M}_{Y^*}(\chi_j)(x, \cdot) = 0, \quad \chi_j(x, \cdot) \text{ } Y\text{-periodic}, \end{cases} \quad (5.17)$$

and χ_0 is the solution of

$$\begin{cases} - \sum_{i,k=1}^n \frac{\partial}{\partial y_i} \left(a_{ik}(x, y) \frac{\partial \chi_0(x, y)}{\partial y_k} \right) = 0 & \text{in } Y^*, \\ \sum_{i,k=1}^n a_{ik}(x, y) \frac{\partial \chi_0(x, y)}{\partial y_k} n_i = g(x, y) & \text{on } \partial S, \\ \mathcal{M}_{Y^*}(\chi_0)(x, \cdot) = 0, \quad \chi_0(x, \cdot) \text{ } Y\text{-periodic}. \end{cases} \quad (5.18)$$

Proof. The proof is straightforward once the existence and uniqueness of the “correctors” χ_i , ($i = 1, \dots, n$) and χ_0 , is shown. For all of them, this follows from the Lax-Milgram theorem. The case of system (5.18), which takes care of the non homogeneous Neumann boundary condition, requires special attention, since to apply the Lax-Milgram theorem a compatibility condition is needed. This condition is $\int_{\partial S} g(x, y) d\sigma(y) = 0$ for a.e. $x \in \Omega$, and it is satisfied due to convergences (5.8) and Remark 4.8. \square

The classical homogenization result for problems with holes is recovered here (without extension operators and with no condition that the holes do not intersect the outer boundary).

Theorem 5.6. *The homogenized formulation associated with Theorem 5.3 is*

$$\begin{cases} -\operatorname{div}(A^0 \nabla u_0) = \frac{|Y^*|}{|Y|} f + \frac{|\partial S|}{|Y|} G - \frac{|\partial S|}{|Y|} \operatorname{div}(\mathcal{M}_{\partial S}(y_{\mathcal{M}} g)) \\ \quad + \frac{|Y^*|}{|Y|} \operatorname{div} \mathcal{M}_{Y^*}(A(x, \cdot) \nabla_y \chi_0(x, \cdot)) \quad \text{in } \Omega, \\ u_0 = 0 \quad \text{on } \partial\Omega, \text{ where } \chi_0 \text{ is given by (5.18).} \end{cases} \quad (5.19)$$

The homogenized matrix $A^0 = (a_{ij}^0)_{1 \leq i, j \leq n}$ is elliptic and defined by

$$a_{ij}^0 = \mathcal{M}_{Y^*} \left(a_{ij} - \sum_{k=1}^n a_{ik} \frac{\partial \chi_j}{\partial y_k} \right) = \mathcal{M}_{Y^*}(a_{ij}) - \mathcal{M}_{Y^*} \left(\sum_{k=1}^n a_{ik} \frac{\partial \chi_j}{\partial y_k} \right). \quad (5.20)$$

where χ_j ($j = 1, \dots, n$) is defined by (5.17).

Proof. Inserting formula (5.16) into (5.10), and taking $\Phi = 0$ gives the result. \square

In the case of a periodic g_ε (see Remark 4.11 for the notations), arising from \mathbf{g} in $L^2(\partial S)$, the two cases give different results. In case 1, $g = 0$, $G = \mathcal{M}_{\partial S}(\mathbf{g})$, so that $\chi_0 = 0$ and the limit problem is

$$\begin{cases} -\operatorname{div}(A^0 \nabla u_0) = \frac{|Y^*|}{|Y|} f + \frac{|\partial S|}{|Y|} \mathcal{M}_{\partial S}(\mathbf{g}) \quad \text{in } \Omega, \\ u_0 = 0 \quad \text{on } \partial\Omega. \end{cases}$$

In case 2, $g = \mathbf{g}$ and $G = 0$, so the limit problem is

$$\begin{cases} -\operatorname{div}(A^0 \nabla u_0) = \frac{|Y^*|}{|Y|} f + \frac{|Y^*|}{|Y|} \operatorname{div} \left(\mathcal{M}_{Y^*}(A(x, \cdot) \nabla_y \chi_0(x, \cdot)) \right) \quad \text{in } \Omega, \\ u_0 = 0 \quad \text{on } \partial\Omega, \end{cases}$$

with χ_0 given by (5.18) associated with \mathbf{g} .

One can remark that in the latter case, if A does not depend on x , χ_0 itself is independent of x . Then, the second term in the right-hand side of the homogenized problem vanishes and \mathbf{g} does not contribute to the limit problem, i.e. the limit equation is the same as that of the homogeneous Dirichlet-Neumann case.

5.2 Convergence of the energy and correctors for the Dirichlet-Neumann problem

Actually, a convergence result stronger than (5.9) (iii) holds for the sequences of solutions $\{u_\varepsilon\}$ of problem (5.1), under stronger assumptions on the data. It is based on the convergence of the energy for this problem.

Proposition 5.7. *Assume that hypotheses of Theorem 5.3 are satisfied. Moreover, assume that the convergences in (5.8) are strong, i.e.,*

$$\begin{aligned} \mathcal{T}_\varepsilon^b(g_\varepsilon) &\rightarrow g \quad \text{strongly in } L^2(\Omega \times \partial S), \\ \frac{1}{\varepsilon} \mathcal{M}_{\partial S}(\mathcal{T}_\varepsilon^b(g_\varepsilon)) &\rightarrow G \quad \text{strongly in } L^2(\mathcal{V}_1(\Omega)). \end{aligned} \quad (5.21)$$

Then

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon \, dx = \frac{1}{|Y|} \int_{\Omega \times Y^*} A [\nabla u_0 + \nabla_y \hat{u}_0] [\nabla u_0 + \nabla_y \hat{u}_0] \, dx \, dy \quad (5.22)$$

and the following strong convergence holds:

$$\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \rightarrow \nabla u_0 + \nabla_y \hat{u}_0 \quad \text{strongly in } L^2(\Omega \times Y^*). \quad (5.23)$$

Moreover,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Lambda_\varepsilon^*} |\nabla u_\varepsilon|^2 \, dx = 0. \quad (5.24)$$

To prove this proposition, we will make use of the following classical result:

Lemma 5.8. *Let $\{D_\varepsilon\}$ be a sequence of $n \times n$ matrix fields in $M(\alpha, \beta, \mathcal{O})$ for some open set \mathcal{O} , such that $D_\varepsilon \rightarrow D$ a.e. on \mathcal{O} (or more generally, in measure in \mathcal{O}). If the sequence $\{\zeta_\varepsilon\}$ converges weakly to ζ in $L^2(\mathcal{O})^n$, then*

$$\liminf_{\varepsilon \rightarrow 0} \int_{\mathcal{O}} D_\varepsilon \zeta_\varepsilon \zeta_\varepsilon \, dx \geq \int_{\mathcal{O}} D \zeta \zeta \, dx. \quad (5.25)$$

Furthermore, if

$$\limsup_{\varepsilon \rightarrow 0} \int_{\mathcal{O}} D_\varepsilon \zeta_\varepsilon \zeta_\varepsilon \, dx \leq \int_{\mathcal{O}} D \zeta \zeta \, dx, \quad (5.26)$$

then

$$\int_{\mathcal{O}} D \zeta \zeta \, dx = \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{O}} D_\varepsilon \zeta_\varepsilon \zeta_\varepsilon \, dx \quad \text{and} \quad \zeta_\varepsilon \rightarrow \zeta \quad \text{strongly in } L^2(\mathcal{O})^n. \quad (5.27)$$

Proof of Proposition 5.7. By Proposition 2.6 (i) and the ellipticity of A^ε ,

$$\begin{aligned} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(A^\varepsilon) \mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) dx dy &= \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon dx - \int_{\Lambda_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon dx \\ &\leq \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon dx. \end{aligned} \quad (5.28)$$

Going back to (5.3), this gives successively

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(A^\varepsilon) \mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) dx dy &\leq \limsup_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon dx \\ &= \limsup_{\varepsilon \rightarrow 0} \left(\int_{\Omega_\varepsilon^*} f u_\varepsilon dx + \int_{\partial S_\varepsilon \cap \Omega} g_\varepsilon u_\varepsilon d\sigma(x) \right). \end{aligned}$$

But, by the three convergences of (5.9) together with formula (4.15) of Proposition 4.10, it follows that $\int_{\Omega_\varepsilon^*} f u_\varepsilon dx \rightarrow \frac{|Y^*|}{|Y|} \int_{\Omega} f u_0 dx$, while

$$\begin{aligned} \int_{\partial S_\varepsilon \cap \Omega} g_\varepsilon u_\varepsilon d\sigma(x) &\rightarrow \frac{|\partial S|}{|Y|} \int_{\Omega} \mathcal{M}_{\partial S}(y_{\mathcal{M}} g) \cdot \nabla u_0 dx \\ &\quad + \frac{|\partial S|}{|Y|} \int_{\Omega} G u_0 dx + \frac{1}{|Y|} \int_{\Omega \times \partial S} \widehat{u}_0 g dx d\sigma(y). \end{aligned}$$

Confronting this with (5.10) where $\Psi = u_0$ and $\Phi = \widehat{u}_0$, proves (5.22). Applying Lemma 5.8 with $D_\varepsilon = \mathcal{T}_\varepsilon^*(A^\varepsilon)$, $\zeta_\varepsilon = \mathcal{T}_\varepsilon^*(\nabla u_\varepsilon)$, yields convergences (5.22) and (5.23). Now, using (5.28) again, gives

$$\lim_{\varepsilon \rightarrow 0} \int_{\Lambda_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon dx = 0,$$

hence (5.24) by ellipticity. \square

We can now address the question of correctors. In the case where $A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right)$, by using extension operators and under the hypothesis that the holes do not intersect $\partial\Omega$, the following corrector result was proved in [19] (Corollary 2.2):

$$\left\| \nabla u_\varepsilon - \nabla u_0 - \sum_{i=1}^n \frac{\partial u_0}{\partial x_i} \nabla_y \chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\}_Y \right) \right\|_{L^1(\Omega_\varepsilon^*)} \rightarrow 0. \quad (5.29)$$

Using the unfolding method, we now give a general corrector result, of which convergence (5.29) is a simple corollary. As in the case of fixed domains (cf. [10]), this corrector result is a direct consequence of the strong convergence (5.23).

Theorem 5.9. *Under the hypotheses of Proposition 5.7, as $\varepsilon \rightarrow 0$, the following strong convergence holds:*

$$\|\nabla u_\varepsilon - \nabla u_0 - \sum_{i=1}^n \mathcal{U}_\varepsilon \left(\frac{\partial u_0}{\partial x_i} \right) \mathcal{U}_\varepsilon^* (\nabla_y \chi_i) - \mathcal{U}_\varepsilon^* (\nabla_y \chi_0)\|_{L^2(\Omega_\varepsilon^*)} \rightarrow 0. \quad (5.30)$$

In the case where the matrix field A does not depend on x , the following corrector result holds:

$$\|u_\varepsilon - u_0 - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left(\frac{\partial u_0}{\partial x_i} \right) \chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\}_Y \right) - \varepsilon \chi_0 \left(\left\{ \frac{\cdot}{\varepsilon} \right\}_Y \right)\|_{H^1(\Omega_\varepsilon^*)} \rightarrow 0. \quad (5.31)$$

Proof. By construction, for $i = 1, \dots, n$, the function χ_i belongs to $L^\infty(\Omega; H^1(Y^*))$. Due to convergences (5.24) and (5.23), Proposition 2.13 (iv) gives

$$\|\nabla u_\varepsilon - \mathcal{U}_\varepsilon^* (\nabla u_0 + \nabla_y \hat{u}_0)\|_{L^2(\Omega_\varepsilon^*)} \rightarrow 0. \quad (5.32)$$

By Proposition 2.11 and (5.16) this implies

$$\|\nabla u_\varepsilon - \nabla u_0 - \sum_{i=1}^n \mathcal{U}_\varepsilon^* \left(\frac{\partial u_0}{\partial x_i} \nabla_y \chi_i \right) - \mathcal{U}_\varepsilon^* (\nabla_y \chi_0)\|_{L^2(\Omega_\varepsilon^*)} \rightarrow 0, \quad (5.33)$$

hence (5.30) follows directly from formula (2.11) and Proposition 1.7 (here, $\nabla_y \chi_i$ is extended by 0 in $\Omega \times S$).

Convergence (5.31) is a consequence of (5.30), and of multiple applications of the following lemma with $\alpha = 1, \frac{\partial u_0}{\partial x_i}$ (extended by 0 outside Ω), $\beta = \chi_i$ and $\beta = \nabla_y \chi_i$ (both extended by 0 in the holes). \square

Lemma 5.10. *There is a constant C such that, for α in $L^2(\mathbb{R}^n)$ and β in $L^2(Y)$,*

$$\left\| \mathcal{Q}_\varepsilon(\alpha) \beta \left(\left\{ \frac{\cdot}{\varepsilon} \right\}_Y \right) \right\|_{L^2(\mathbb{R}^n)} \leq C \|\alpha\|_{L^2(\mathbb{R}^n)} \|\beta\|_{L^2(Y)}, \quad (5.34)$$

and moreover,

$$\left\| \mathcal{U}_\varepsilon(\alpha) \beta \left(\left\{ \frac{\cdot}{\varepsilon} \right\}_Y \right) - \mathcal{Q}_\varepsilon(\alpha) \beta \left(\left\{ \frac{\cdot}{\varepsilon} \right\}_Y \right) \right\|_{L^2(\mathbb{R}^n)} \rightarrow 0. \quad (5.35)$$

Proof. For the proof of (5.34), we refer the reader to [21]. Convergence (5.35) follows from (5.10) and from Propositions 1.6 (iv) and 1.7, using the strong convergence of $\mathcal{Q}_\varepsilon(\alpha)$ to α in $L^2(\mathbb{R}^n)$ (see [10]). \square

5.3 Homogenization of the Neumann problem

To homogenize the Neumann problem (5.2), the same method applies. We state the results without detailing the proofs. The condition on b_ε is the following:

$$\left\{ \begin{array}{l} \text{There are two positive constants } c_0 \text{ and } C_0 \text{ and } b \in L^\infty(\Omega \times Y^*) \text{ such that,} \\ c_0 \leq b_\varepsilon(x) \leq C_0 \text{ a.e. } x \in \Omega_\varepsilon^* \text{ and } \mathcal{T}_\varepsilon^*(b_\varepsilon) \rightarrow b \text{ in measure (or a.e.) in } \Omega \times Y^*. \end{array} \right. \quad (5.36)$$

By Corollary 2.15, the last condition of (5.36) is equivalent to

$$\|b_\varepsilon - b\|_{L^p(\widehat{\Omega}_\varepsilon^*)} \rightarrow 0 \quad \text{for some (or every!) } p \in [1, \infty).$$

Theorem 5.11. (*Unfolded formulation for (5.2)*). *Let u_ε be the solution of the problem (5.4). Assume that (5.7), (5.8) and (5.36) hold. Assume furthermore that g_ε vanishes outside of $\widehat{\partial S}_\varepsilon$. Then, there exist u in $H^1(\Omega)$ and \widehat{u} in $L^2(\Omega; H_{per}^1(Y^*))$, such that*

$$\begin{aligned} (i) \quad & \mathcal{T}_\varepsilon^*(u_\varepsilon) \text{ converges to } u \text{ weakly in } L^2(\Omega; H^1(Y^*)) \text{ and strongly in } L_{loc}^2(\Omega; H^1(Y^*)) \\ (ii) \quad & \mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \rightharpoonup \nabla u + \nabla_y \widehat{u} \quad \text{weakly in } L^2(\Omega \times Y^*), \end{aligned} \quad (5.37)$$

and the pair (u, \widehat{u}) is the unique solution of the problem

$$\left\{ \begin{array}{l} u \in H^1(\Omega), \widehat{u} \in L^2(\Omega; H_{per}^1(Y^*)) \text{ with } \mathcal{M}_{Y^*}(\widehat{u}) = 0 \text{ for a.e. } x \in \Omega, \\ \frac{1}{|Y|} \int_{\Omega \times Y^*} A(x, y) [\nabla u(x) + \nabla_y \widehat{u}(x, y)] [\nabla \Psi(x) + \nabla_y \Phi(x, y)] \, dx dy \\ \quad + \frac{|Y^*|}{|Y|} \int_{\Omega} \mathcal{M}_{Y^*}(b)(x) u(x) \Psi(x) \, dx \\ = \frac{|Y^*|}{|Y|} \int_{\Omega} f(x) \Psi(x) \, dx + \frac{|\partial S|}{|Y|} \int_{\Omega} \mathcal{M}_{\partial S}(y_{\mathcal{M}g})(x) \cdot \nabla \Psi(x) \, dx \\ \quad + \frac{|\partial S|}{|Y|} \int_{\Omega} G(x) \Psi(x) \, dx + \frac{1}{|Y|} \int_{\Omega \times \partial S} g(x, y) \Phi(x, y) \, dx \, d\sigma(y), \\ \forall \Psi \in H^1(\Omega), \forall \Phi \in L^2(\Omega; H_{per}^1(Y^*)). \end{array} \right. \quad (5.38)$$

Proof. The a priori estimate

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon^*)} \leq C, \quad (5.39)$$

follows directly from the variational formulation (5.4) and Proposition 4.3. The remainder of the proof is the exact analog of the proof of Theorem 5.3, making use of Theorem 3.12 instead of Theorem 3.13. \square

The next results are the equivalent of Proposition 5.5 and Theorem 5.6 (with obvious modifications in the proofs).

Proposition 5.12. *The function \hat{u} in Theorem 5.11 is given in terms of u by*

$$\hat{u}(x, y) = \sum_{i=1}^n \frac{\partial u}{\partial x_i}(x) \chi_i(x, y) + \chi_0,$$

where the corrector functions χ_j , ($j = 0, \dots, n$) are given, as before, by (5.17) and (5.18).

Theorem 5.13. *(Standard homogenization for Neumann problem). Let u_ε be the solution of problem (5.4) and suppose that the hypotheses of Theorem 5.11 are satisfied. Then u is the solution in $H^1(\Omega)$ of the homogenized problem*

$$\begin{cases} -\operatorname{div} (A^0 \nabla u) + \frac{|Y^*|}{|Y|} \mathcal{M}_{Y^*}(b) u = \frac{|Y^*|}{|Y|} f - \operatorname{div} \mathcal{G} & \text{in } \Omega, \\ A^0 \nabla u \cdot n = \mathcal{G} \cdot n & \text{on } \partial\Omega, \end{cases}$$

where

$$\mathcal{G}(x) \doteq \frac{|\partial S|}{|Y|} (G + \mathcal{M}_{\partial S}(y_{\mathcal{M}} g)(x)) - \frac{|Y^*|}{|Y|} \mathcal{M}_{Y^*}(A(x, \cdot) \nabla_y \chi_0(x, \cdot)) \quad \text{in } \Omega.$$

The matrix field A^0 is the same as that defined in Theorem 5.6.

Remark 5.14. *In this problem, a strange phenomenon occurs: the non homogeneous Neumann conditions on the boundary of the holes inside Ω (actually supported inside $\hat{\Omega}_\varepsilon$) contribute to a non homogeneous Neumann condition on the outer boundary $\partial\Omega$ in the limit problem through the term \mathcal{G} . This phenomenon was also observed in the context of Γ -convergence in [6].*

In the case of periodic a g_ε derived from \mathbf{g} be in $L^2(\partial S)$, the two cases of Remark 4.11 give different results.

In case 3, the limit problem is (since $g = 0$, $\chi_0 = 0$ so $\mathcal{G} = 0$)

$$\begin{cases} -\operatorname{div} (A^0 \nabla u) + \frac{|Y^*|}{|Y|} \mathcal{M}_{Y^*}(b) u = \frac{|Y^*|}{|Y|} f + \frac{|\partial S|}{|Y|} \mathcal{M}_{\partial S}(\mathbf{g}) & \text{in } \Omega, \\ A^0 \nabla u \cdot n = 0 & \text{on } \partial\Omega. \end{cases}$$

In case 4 of Remark 4.11, the limit problem is

$$\begin{cases} -\operatorname{div} (A^0 \nabla u) + \frac{|Y^*|}{|Y|} \mathcal{M}_{Y^*}(b) u = \frac{|Y^*|}{|Y|} f - \operatorname{div} \mathcal{G} & \text{in } \Omega, \\ A^0 \nabla u \cdot n = \mathcal{G} \cdot n & \text{on } \partial\Omega, \end{cases}$$

with $\mathcal{G}(x) \doteq \frac{|\partial S|}{|Y|} \mathcal{M}_{\partial S}(y_{\mathcal{M}} \mathbf{g})(x) - \frac{|Y^*|}{|Y|} \mathcal{M}_{Y^*}(A(x, \cdot) \nabla_y \chi_0(x, \cdot))$ in Ω .

In this last case, the holes induce a non homogeneous Neumann condition at the limit.

5.3.1 Convergence of the energy and correctors for the Neumann case

The following result is the equivalent of Proposition 5.7.

Proposition 5.15. *(Convergence of the energy for problem (5.4)). Assume also that*

$$\begin{aligned}\mathcal{T}_\varepsilon^b(g_\varepsilon) &\rightarrow g \quad \text{strongly in } L^2(\Omega \times \partial S), \\ \frac{1}{\varepsilon} \mathcal{M}_{\partial S}(\mathcal{T}_\varepsilon^b(g_\varepsilon)) &\rightarrow G \quad \text{strongly in } L^2(\Omega).\end{aligned}\tag{5.40}$$

Then

$$\begin{aligned}\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} \left(A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon + b_\varepsilon u_\varepsilon^2 \right) dx &= \frac{1}{|Y|} \int_{\Omega \times Y^*} A[\nabla u + \nabla_y \hat{u}] [\nabla u + \nabla_y \hat{u}] dx dy \\ &\quad + \frac{|Y^*|}{|Y|} \int_{\Omega} \mathcal{M}_{Y^*}(b) u^2 dx,\end{aligned}\tag{5.41}$$

and

$$\lim_{\varepsilon \rightarrow 0} \left(\int_{\Lambda_\varepsilon^*} |\nabla u_\varepsilon|^2 dx + \int_{\Lambda_\varepsilon^*} u_\varepsilon^2 dx \right) = 0.\tag{5.42}$$

Moreover, one has the following convergences:

$$\begin{aligned}\mathcal{T}_\varepsilon^*(u_\varepsilon) &\rightarrow u \quad \text{strongly in } L^2(\Omega \times Y^*), \\ \mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) &\rightarrow \nabla u + \nabla_y \hat{u} \quad \text{strongly in } L^2(\Omega \times Y^*).\end{aligned}\tag{5.43}$$

Proof. The proof of (5.41) is similar to that in Proposition 5.7. For that of (5.43) and (5.42), Lemma 5.8 is applied to the $(n+1) \times (n+1)$ matrix D_ε and the $n+1$ vector ξ_ε

$$D_\varepsilon = \begin{pmatrix} & & 0 \\ & \mathcal{T}_\varepsilon^*(A^\varepsilon) & \vdots \\ 0 & \dots & 0 & \mathcal{T}_\varepsilon^*(b_\varepsilon) \end{pmatrix}, \quad \xi_\varepsilon = \begin{pmatrix} \mathcal{T}_\varepsilon^*\left(\frac{\partial u_\varepsilon}{x_1}\right) \\ \vdots \\ \mathcal{T}_\varepsilon^*\left(\frac{\partial u_\varepsilon}{x_n}\right) \\ \mathcal{T}_\varepsilon^*(u_\varepsilon) \end{pmatrix}.$$

□

As a consequence, we have

Theorem 5.16. *As $\varepsilon \rightarrow 0$,*

$$\begin{aligned}(i) \quad &\|u_\varepsilon - u\|_{L^2(\Omega_\varepsilon^*)} \rightarrow 0, \\ (ii) \quad &\|\nabla u_\varepsilon - \nabla u_0 - \sum_{i=1}^n \mathcal{U}_\varepsilon\left(\frac{\partial u_0}{\partial x_i}\right) \mathcal{U}_\varepsilon^*(\nabla_y \chi_i) - \mathcal{U}_\varepsilon^*(\nabla \chi_0)\|_{L^2(\Omega_\varepsilon^*)} \rightarrow 0.\end{aligned}\tag{5.44}$$

In the case where the matrix field A does not depend on x , the following corrector result holds:

$$\left\| u_\varepsilon - u_0 - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left(\frac{\partial u_0}{\partial x_i} \right) \chi_i \left(\left\{ \frac{\cdot}{\varepsilon} \right\}_Y \right) - \varepsilon \chi_0 \left(\left\{ \frac{\cdot}{\varepsilon} \right\}_Y \right) \right\|_{H^1(\Omega_\varepsilon^*)} \rightarrow 0.$$

Proof. Convergence (i) follows from Corollary 2.15, (5.42) and (5.43). The proof of the other convergences is the same as that of Proposition 5.9. \square

5.4 Multiscales domains mixing composites and perforations

As shown in [10], the periodic unfolding method is particularly well-adapted to multiscales problems (as compared to the use of two-scale convergence in [1], where the scales have to be well-separated). The unfolding methods for fixed domains and for perforated domains, can be combined to consider mixed situations.

Let Y^* be a subset of Y for which Hypothesis (\mathbf{H}_p) (see Section 3.2.1) is satisfied, see Subsection 3.2.1. Let Y_2 be given an open subset of Y^* with Lipschitz boundary and denote $Y^* \setminus Y_2$ by Y_1 . Let Z be another periodicity cell, and ε, δ be two small parameters.

For x in $(\mathbb{R}^n)_\varepsilon^*$, set $A^{\varepsilon\delta}$ be a matrix field defined by

$$A^{\varepsilon\delta}(x) = \begin{cases} A_1 \left(\left\{ \frac{x}{\varepsilon} \right\}_Y \right) & \text{for } \left\{ \frac{x}{\varepsilon} \right\}_Y \in Y_1, \\ A_2 \left(\left\{ \frac{\left\{ \frac{x}{\varepsilon} \right\}_Y}{\delta} \right\}_Z \right) & \text{for } \left\{ \frac{x}{\varepsilon} \right\}_Y \in Y_2, \end{cases}$$

where A_1 is in $M(\alpha, \beta, Y_1)$ and A_2 in $M(\alpha, \beta, Z)$.

With the notation from Section 3, the perforated domain Ω_ε^* (see Figure 8) is defined by (2.3), i.e., $\Omega_\varepsilon^* = \Omega \setminus \overline{S}_\varepsilon$, where $\overline{S}_\varepsilon = \bigcup_{\xi \in \mathcal{G}} \varepsilon(\xi + \overline{B})$. So, Ω_ε^* has ε -periodic holes (the set S_ε) and an ε -periodic set of a composite material corresponding to the set $Y_{2,\varepsilon} = \bigcup_{\xi \in \mathcal{G}} \varepsilon(\xi + Y_2)$.

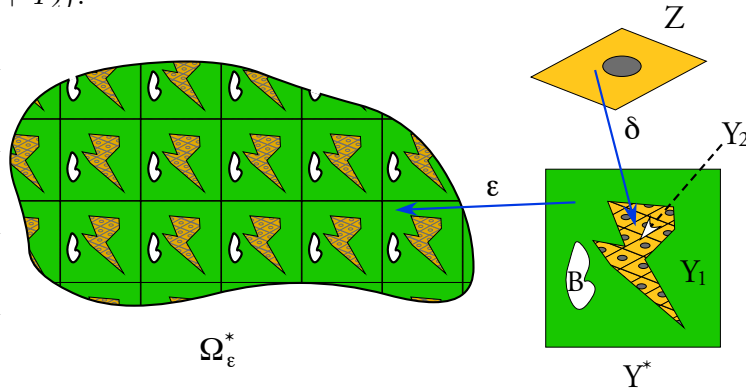


Figure 8. The perforated domain Ω_ε^*

Consider now the problem

$$\int_{\Omega_\varepsilon^*} A^{\varepsilon\delta} \nabla u_{\varepsilon\delta} \nabla v \, dx = \int_{\Omega_\varepsilon^*} f v \, dx, \quad \forall v \in H_0^1(\Omega_\varepsilon^*; \partial\Omega \cap \partial\Omega_\varepsilon^*), \quad (5.45)$$

where f in $L^2(\Omega)$. By the Lax-Milgram theorem, one has existence and uniqueness of $u_{\varepsilon\delta}$ in $H_0^1(\Omega_\varepsilon^*; \partial\Omega \cap \partial\Omega_\varepsilon^*)$ satisfying the estimate

$$\|u_{\varepsilon\delta}\|_{H^1(\Omega_\varepsilon^*)} \leq \frac{1}{\alpha} \|f\|_{L^2(\Omega)}.$$

Unfolding at the scale ε as in Section 5, we have convergences (3.33) for some \widehat{u} in $L^2(\Omega; H_{per}^1(Y^*))$, i.e.,

$$\begin{aligned} \mathcal{T}_\varepsilon^*(u_{\varepsilon\delta}) &\rightarrow u_0 \quad \text{strongly in } L^2(\Omega; H^1(Y^*)), \\ \mathcal{T}_\varepsilon^*(\nabla u_{\varepsilon\delta}) &\rightharpoonup \nabla u_0 + \nabla_y \widehat{u} \quad \text{weakly in } L^2(\Omega \times Y^*). \end{aligned}$$

At this level, we do not see the oscillations at the scale $\varepsilon\delta$. To capture them, we unfold at the scale δ , as in [10], Section 7. To do so, consider the restrictions to the set $\Omega \times Y_2$ defined by

$$v_{\varepsilon\delta}(x, y) \doteq \frac{1}{\varepsilon} \mathcal{T}_\varepsilon^*(\mathcal{R}_\varepsilon^*(u_{\varepsilon\delta}))|_{\Omega \times Y_2}.$$

It is immediate that (up to a subsequence)

$$v_{\varepsilon\delta} \rightharpoonup \widehat{u}|_{\Omega \times Y_2} \quad \text{weakly in } L^2(\Omega; H^1(Y_2)).$$

We now apply to $v_{\varepsilon\delta}$ the unfolding operator \mathcal{T}_δ^y for the variable y , defined by

$$\mathcal{T}_\delta^y(v_{\varepsilon\delta})(x, y, z) = v_{\varepsilon\delta}\left(x, \delta \left\lceil \frac{y}{\delta} \right\rceil_Z + \delta z\right) \quad \text{for } x \in \Omega, \ y \in Y_2 \text{ and } z \in Z,$$

adding so the new variable z .

Obviously, all the estimates and weak convergence properties which were shown for the original unfolding \mathcal{T}_ε still hold for \mathcal{T}_δ^y with x being a mere parameter. Therefore, by simply adapting Theorems 1.3 and 1.4, one has the convergences

$$\begin{aligned} \mathcal{T}_\delta^y(\nabla_y v_{\varepsilon\delta}) &\rightharpoonup \nabla_y \widehat{u} + \nabla_z \widehat{u}_1 \quad \text{weakly in } L^2(\Omega \times Y_2 \times Z), \\ \mathcal{T}_\delta^y\left(\mathcal{T}_\varepsilon^*(\nabla u_{\varepsilon\delta})\right) &\rightharpoonup \nabla u_0 + \nabla_y \widehat{u} + \nabla_z \widehat{u}_1 \quad \text{weakly in } L^2(\Omega \times Y_2 \times Z), \end{aligned}$$

for some \widehat{u}_1 in $L^2(\Omega \times Y_2; H_{per}^1(Z))$.

In summary, we have the following homogenization result for problem (5.45):

Theorem 5.17. *The functions*

$$u_0 \in H_0^1(\Omega), \quad \widehat{u} \in L^2(\Omega; H_{per}^1(Y^*)/\mathbb{R}), \quad \widehat{u}_1 \in L^2(\Omega \times Y_2; H_{per}^1(Z)/\mathbb{R}),$$

are the unique solutions of the variational problem

$$\left\{ \begin{array}{l} \frac{1}{|Y||Z|} \int_{\Omega} \int_{Y_2} \int_Z A_2(z) \left\{ \nabla u_0 + \nabla_y \widehat{u} + \nabla_z \widehat{u}_1 \right\} \left\{ \nabla \Psi + \nabla_y \Phi + \nabla_z \Theta \right\} dx dy dz \\ \quad + \frac{1}{|Y|} \int_{\Omega} \int_{Y_1} A_1(y) \left\{ \nabla u_0 + \nabla_y \widehat{u} \right\} \left\{ \nabla \Psi + \nabla_y \Phi \right\} dx dy = \frac{|Y^*|}{|Y|} \int_{\Omega} f \Psi dx, \\ \forall \Psi \in H_0^1(\Omega), \quad \forall \Phi \in L^2(\Omega; H_{per}^1(Y^*)/\mathbb{R}), \quad \forall \Theta \in L^2(\Omega \times Y_2, H_{per}^1(Z)/\mathbb{R}). \end{array} \right.$$

For the proof, one introduces test functions of the form

$$\Psi(x) + \varepsilon \Psi_1(x) \Phi_1\left(\frac{x}{\varepsilon}\right) + \varepsilon \delta \Psi_2(x) \Phi_2\left(\left\{\frac{x}{\varepsilon}\right\}_Y\right) \Theta_2\left(\frac{1}{\delta} \left\{\frac{x}{\varepsilon}\right\}_Z\right),$$

where Ψ, Ψ_1, Ψ_2 are in $\mathcal{D}(\Omega)$, Φ_1 in $H_{per}^1(Y^*)$, Φ_2 in $\mathcal{D}(Y_2)$ and Θ_2 in $H_{per}^1(Z)$, and proceed as in the preceding section.

Proposition 5.7 extends without any difficulty to the multiscale case.

Proposition 5.18. *The convergence for the energy holds true,*

$$\begin{aligned} \lim_{\varepsilon, \delta \rightarrow 0} \int_{\Omega_{\varepsilon}^*} A^{\varepsilon \delta} \nabla u_{\varepsilon \delta} \nabla u_{\varepsilon \delta} dx \\ = \frac{1}{|Y||Z|} \int_{\Omega} \int_{Y_2} \int_Z A_2(z) \left\{ \nabla u_0 + \nabla_y \widehat{u} + \nabla_z \widehat{u}_1 \right\} \left\{ \nabla u_0 + \nabla_y \widehat{u} + \nabla_z \widehat{u}_1 \right\} dx dy dz \\ + \frac{1}{|Y|} \int_{\Omega} \int_{Y_1} A_1(y) \left\{ \nabla u_0 + \nabla_y \widehat{u} \right\} \left\{ \nabla u_0 + \nabla_y \widehat{u} \right\} dx dy. \end{aligned}$$

Moreover, one has the following strong convergences:

$$\begin{aligned} \mathcal{T}_{\varepsilon}^*(\nabla u_{\varepsilon \delta}) &\rightarrow \nabla u_0 + \nabla_y \widehat{u} \quad \text{strongly in } L^2(\Omega \times Y_1), \\ \mathcal{T}_{\delta}^y(\mathcal{T}_{\varepsilon}^*(\nabla u_{\varepsilon \delta})) &\rightarrow \nabla u_0 + \nabla_y \widehat{u} + \nabla_z \widehat{u}_1 \quad \text{strongly in } L^2(\Omega \times Y_2 \times Z). \end{aligned}$$

Remark 5.19. *In the previous situation Z can be replaced by a perforated subset Z^* , leading to two levels of perforations in the domain Ω . In Theorem 5.17, supposing that Z^* also satisfies hypothesis **(H₂)**, \widehat{u}_1 and Θ belong to $L^2(\Omega \times Y_2, H_{per}^1(Z^*)/\mathbb{R})$ while in the first integral, Z is replaced by Z^* .*

Remark 5.20. *Theorem 5.17 can be extended to the case of any finite number of distinct scales, by a simple reiteration process.*

An example of application of Theorem 5.17 is the case of a reticulated structure with all vertical bars made from a composite material (see Figures 9 and 10).

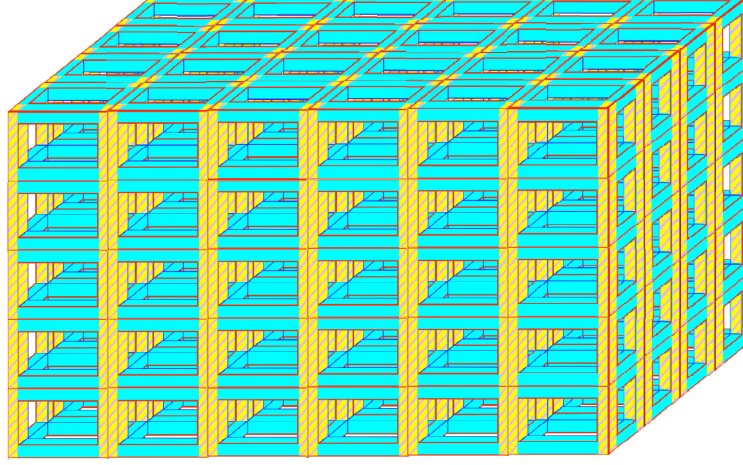


Figure 9. The reticulated structure

In this example, the periodicity cell Y^* is the union of Y_1 , the set of horizontal bars, and Y_2 , the set of vertical bars, so that $Y_1 \cap Y_2 = \emptyset$ and $\bar{Y}^* = \bar{Y}_1 \cup \bar{Y}_2$.

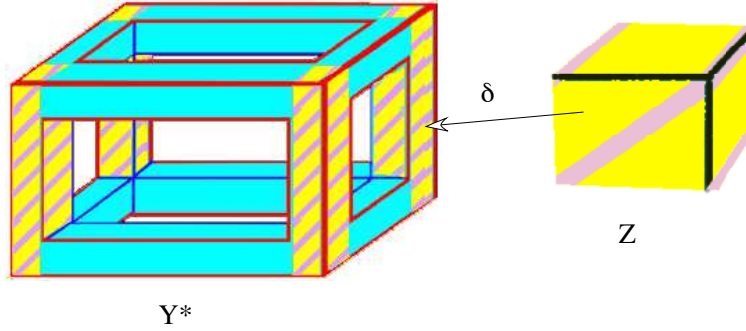


Figure 10. The periodicity cell Y^*

6 Appendix: The macro-micro operators $\mathcal{Q}_\varepsilon^*$ and $\mathcal{R}_\varepsilon^*$ when the reference cell Y is *not* a parallelotop

In this Appendix, just as in Subsection 3.2.2, the macro approximation at the points of Ξ_ε is constructed by an average and is then extended by Q_1 -interpolation in the parallelotops $\varepsilon(\xi + \mathcal{P})$, $\xi \in \Xi_\varepsilon^\mathcal{Y}$. Consequently, the macro-approximation is naturally defined on the set

$$\hat{\Omega}_\varepsilon^\mathcal{P} = \text{interior} \left\{ \bigcup_{\xi \in \Xi_\varepsilon^\mathcal{Y}} \varepsilon(\xi + \bar{\mathcal{P}}) \right\}, \quad (6.1)$$

where \mathcal{P} was introduced in (2.2) and $\Xi_\varepsilon^\mathcal{Y}$ in (3.4). Note that $\hat{\Omega}_\varepsilon^\mathcal{P}$ is *not necessarily included in* Ω (for example, this can occur if Y is offset with respect to \mathcal{P}).

Definition 6.1. *The operator $\mathcal{Q}_\varepsilon^* : L^p(\Omega_\varepsilon^*) \mapsto W^{1,\infty}(\widehat{\Omega}_\varepsilon^{\mathcal{P}})$, for $p \in [1, +\infty]$, is defined as follows*

$$\mathcal{Q}_\varepsilon^*(\phi)(\varepsilon\xi) = \frac{1}{|Y^*|} \int_{Y^*} \phi(\varepsilon\xi + \varepsilon z) dz = \mathcal{M}_{\varepsilon\xi + \varepsilon Y^*}(\phi) \quad \text{for all } \xi \text{ in } \Xi_\varepsilon^{\mathcal{Y}} + \mathcal{K},$$

and for every $x \in \widehat{\Omega}_\varepsilon^{\mathcal{P}}$,

$$\begin{cases} \mathcal{Q}_\varepsilon^*(\phi)(x) \text{ is the } Q_1 \text{ interpolate of the values of } \mathcal{Q}_\varepsilon^*(\phi) \text{ at the vertices} \\ \text{of the parallelotop } \varepsilon \left[\frac{x}{\varepsilon} \right]_Y + \varepsilon \mathcal{P}. \end{cases} \quad (6.2)$$

We can easily check that the results given in Propositions 3.6 and 3.7 are still valid replacing $\widehat{\Omega}_\varepsilon^{\mathcal{Y}}$ by $\widehat{\Omega}_\varepsilon^{\mathcal{P}}$.

However, the remainder $\mathcal{R}_\varepsilon^*(\phi) = \phi - \mathcal{Q}_\varepsilon^*(\phi)$ is now defined only on $\Omega_\varepsilon^* \cap \widehat{\Omega}_\varepsilon^{\mathcal{P}}$.

To go further, we need to estimate the L^p -norm of $\mathcal{R}_\varepsilon^*(\phi)$ only in terms of the gradient of ϕ . But this is not always possible on $\Omega_\varepsilon^* \cap \widehat{\Omega}_\varepsilon^{\mathcal{P}}$ (since this set is not always a union of cells of the type $\varepsilon(\xi + Y^*)$). We are therefore led to consider a subset of $\Omega_\varepsilon^* \cap \widehat{\Omega}_\varepsilon^{\mathcal{P}}$ defined as a union of cells of the type $\varepsilon(\xi + Y^*)$ included in Ω_ε^* . Since this subset will play the same role as $\widehat{\Omega}_\varepsilon^{**}$ in the previous case (see (3.22)), we will still denote it $\widehat{\Omega}_\varepsilon^{**}$.

To give its new definition, we use the facts that the parallelotop \mathcal{P} satisfies the paving property (2.1) and that Y is a bounded domain. Therefore, the latter can be covered by a finite union of \mathcal{G} -translates of \mathcal{P} , more precisely, there exists some $k > 0$ and

$$b'_1, b'_2, \dots, b'_k \text{ in } \mathcal{G}$$

such that (see Figure A1)

$$\widehat{\mathcal{P}} \doteq \text{interior} \left(\bigcup_{i=1}^k (b'_i + \overline{\mathcal{P}}) \right) \text{ is connected and } Y \subset \widehat{\mathcal{P}}. \quad (6.3)$$

Now set

$$\Xi'_\varepsilon \doteq \left\{ \xi \in \Xi_\varepsilon \mid \varepsilon(\xi + b'_j + \overline{\mathcal{P}} \cup \overline{\mathcal{Y}}) \subset \Omega, \quad \forall j = 1, \dots, k \right\}, \quad (6.4)$$

$$\widehat{\Omega}'_\varepsilon \doteq \text{interior} \left\{ \bigcup_{\xi \in \Xi'_\varepsilon} \varepsilon(\xi + \overline{Y}) \right\}, \quad (6.5)$$

and

$$\widehat{\Omega}_\varepsilon^{**} \doteq \Omega_\varepsilon^* \cap \widehat{\Omega}'_\varepsilon = \text{interior} \left\{ \bigcup_{\xi \in \Xi'_\varepsilon} \varepsilon(\xi + \overline{Y^*}) \right\}. \quad (6.6)$$

Since, by construction, $\widehat{\Omega}'_\varepsilon$ is included in $\widehat{\Omega}_\varepsilon^{\mathcal{P}}$, one has $\widehat{\Omega}_\varepsilon^{**} \subset \Omega_\varepsilon^* \cap \widehat{\Omega}_\varepsilon^{\mathcal{P}}$.

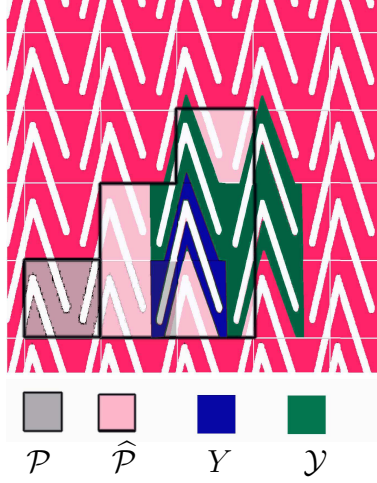


Figure A1. The sets $\mathcal{P}, \widehat{\mathcal{P}}, Y$ and \mathcal{Y}

Note that this is a generalization of the situation of the previous section since for $Y = \mathcal{P}$, the three sets $\widehat{\Omega}'_\varepsilon, \widehat{\Omega}^\mathcal{P}_\varepsilon$ and $\widehat{\Omega}^\mathcal{Y}_\varepsilon$ coincide. In general, the following inclusions hold:

$$\widehat{\Omega}^{**}_\varepsilon \subset \widehat{\Omega}'_\varepsilon \subset \widehat{\Omega}^\mathcal{P}_\varepsilon, \quad \widehat{\Omega}'_\varepsilon \subset \widehat{\Omega}^\mathcal{Y}_\varepsilon \subset \widehat{\Omega}_\varepsilon \subset \Omega.$$

Usually, the open set $\widehat{\Omega}^\mathcal{P}_\varepsilon$ is not included in $\widehat{\Omega}^\mathcal{Y}_\varepsilon$ or $\widehat{\Omega}^Y_\varepsilon$.

All the following numbers, which measure the width of the boundary layer for each subset, are bounded above by a constant multiple of ε :

$$\sup_{x \in \partial \widehat{\Omega}_\varepsilon} \text{dist}(x, \partial \Omega), \quad \sup_{x \in \partial \widehat{\Omega}^\mathcal{Y}_\varepsilon} \text{dist}(x, \partial \Omega), \quad \sup_{x \in \partial \widehat{\Omega}^\mathcal{P}_\varepsilon} \text{dist}(x, \partial \Omega), \quad \sup_{x \in \partial \widehat{\Omega}^\mathcal{R}_\varepsilon} \text{dist}(x, \partial \Omega). \quad (6.7)$$

They depend on Ω, Y, \mathcal{P} and on the k vectors introduced in (6.3). If ω is a relatively compact open subset of Ω then, for ε small enough, it is included in all the open sets $\widehat{\Omega}_\varepsilon, \widehat{\Omega}^\mathcal{Y}_\varepsilon, \widehat{\Omega}^\mathcal{P}_\varepsilon$ and $\widehat{\Omega}'_\varepsilon$.

As in (3.22), for ϕ in $L^p(\Omega_\varepsilon^*)$, p in $[1, +\infty]$ we write

$$\phi = \mathcal{Q}_\varepsilon^*(\phi) + \mathcal{R}_\varepsilon^*(\phi), \quad \text{a.e. in } \widehat{\Omega}_\varepsilon^{**}, \quad (6.8)$$

with $\widehat{\Omega}_\varepsilon^{**}$ defined here by (6.6). On this set, one can get the L^p -estimate for $\mathcal{R}_\varepsilon^*(\phi)$. The result below is similar to Proposition 3.8.

Proposition 6.2. *There is a constant C independent of ε such that for every ϕ in $W^{1,p}(\Omega_\varepsilon^*)$,*

- (i) $\|\mathcal{R}_\varepsilon^*(\phi)\|_{L^p(\widehat{\Omega}_\varepsilon^{**})} \leq \varepsilon C \|\nabla \phi\|_{L^p(\Omega_\varepsilon^*)},$
- (ii) $\|\nabla \mathcal{R}_\varepsilon^*(\phi)\|_{L^p(\widehat{\Omega}_\varepsilon^{**})} \leq C \|\nabla \phi\|_{L^p(\Omega_\varepsilon^*)}.$

Proof. Inequality (ii) is immediate from Proposition 3.7.

To prove inequality (i), let ϕ be in $W^{1,p}(\Omega_\varepsilon^*)$. In this setting, inequality (3.24) still holds for every $\xi \in \Xi'_\varepsilon$ and implies for every $j = 1, \dots, k$,

$$\|\phi - \mathcal{Q}_\varepsilon^*(\phi)(\varepsilon\xi)\|_{L^p(\varepsilon(\xi+Y^*\cap(b'_j+\mathcal{P})))} \leq C\varepsilon\|\nabla\phi\|_{L^p(\varepsilon(\xi+Y^*))}. \quad (6.9)$$

For every ξ in $\Xi_\varepsilon^\mathcal{Y}$ and such that $\varepsilon(\xi + \overline{\mathcal{P}}) \subset \Omega$, rescaling inequality (3.21) and using the definition of $\mathcal{Q}_\varepsilon^*(\phi)$ give

$$\|\mathcal{Q}_\varepsilon^*(\phi) - \mathcal{Q}_\varepsilon^*(\phi)(\varepsilon\xi)\|_{L^p(\varepsilon\xi+\varepsilon\mathcal{P})} \leq C\varepsilon\|\nabla\phi\|_{L^p(\varepsilon(\xi+\mathcal{Y}^*))},$$

hence, for every ξ in Ξ'_ε and for every $j = 1, \dots, k$,

$$\|\mathcal{Q}_\varepsilon^*(\phi) - \mathcal{Q}_\varepsilon^*(\phi)(\varepsilon\xi)\|_{L^p(\varepsilon(\xi+Y^*\cap(b'_j+\mathcal{P})))} \leq C\varepsilon\|\nabla\phi\|_{L^p(\varepsilon(\xi+b'_j+\mathcal{Y}^*))}, \quad (6.10)$$

Combining (6.10) and (6.9) and summing over j gives

$$\|\phi - \mathcal{Q}_\varepsilon^*(\phi)\|_{L^p(\varepsilon(\xi+Y^*))} = \|\mathcal{R}_\varepsilon^*(\phi)\|_{L^p(\varepsilon(\xi+Y^*))} \leq C\varepsilon \sum_{j=1,\dots,k} \|\nabla\phi\|_{L^p(\varepsilon(\xi+b'_j+\mathcal{Y}^*))}.$$

Finally, using definition (6.6) of $\widehat{\Omega}_\varepsilon^{**}$, inequality (i) follows by summation over Ξ'_ε . \square

References

- [1] G. Allaire and M. Briane, Multiscale convergence and reiterated homogenization, Proc. Royal Soc. Edinburgh, 126 A(1996), 297–342.
- [2] G. Allaire, A. Damlamian and U. Hornung, Two-scale convergence on periodic surfaces and applications, in *Proceedings of the International Conference on Mathematical Modelling of Flow through Porous Media, 1995*, A. Bourgeat and C. Carasso eds., World Scientific Publications, Singapore, 1996.
- [3] G. Allaire and F. Murat, Homogenization of the homogeneous Neumann problem with nonisolated holes, including Appendix with A.K. Nandakumar, Asymptotic Analysis 7 (1993), 81–95.
- [4] N.S. Bakhvalov and G. Panasenko, *Homogenization: Averaging Processes in Periodic Media*, Kluwer, Dordrecht, 1989.
- [5] M. Briane, A. Damlamian and P. Donato, H-convergence in perforated domains, in Nonlinear Partial Diff. Eqs and Their Applications, Collège de France seminar vol. XIII, D. Cioranescu and J.-L. Lions eds., 62–100, Pitman Research Notes in Mathematics Series, Longman, New York 1998.

- [6] V. Chiadò Piat and A. Piatnitski, Γ -convergence approach to variational problems in perforated domains with Fourier boundary conditions, ESAIM: COCV, 16 (2010), 148–175.
- [7] G. A. Chechkin, Homogenization in perforated domains, in *Topics on concentration phenomena and problems with multiple scales*, Lect. Notes Unione Mat. Ital., 2, Springer, Berlin, 2006.
- [8] D. Cioranescu, A. Damlamian and G. Griso, Periodic unfolding and homogenization, C. R. Acad. Sci. Paris, Série 1, 335 (2002), 99–104.
- [9] D. Cioranescu, A. Damlamian and G. Griso, The Stokes problem in perforated domains by the periodic unfolding method. in *Proceedings Conference New Trends in Continuum Mechanics*, Ed. M. Suliciu, Theta, Bucarest, 2005, 67–80.
- [10] D. Cioranescu, A. Damlamian and G. Griso, The periodic unfolding method in homogenization, SIAM J. of Math. Anal. Vol. 40, 4 (2008), 1585–1620.
- [11] D. Cioranescu, A. Damlamian, G. Griso and D. Onofrei, The periodic unfolding method for perforated domains and Neumann sieve models, J. Math. Pures Appl., 89 (2008), 248–277.
- [12] D. Cioranescu and P. Donato, Homogénéisation du problème de Neumann non homogène dans des ouverts perforés, Asymptotic Analysis, 1 (1988), 115–138.
- [13] D. Cioranescu, P. Donato and R. Zaki, The periodic unfolding method in perforated domains, Portugaliae Mathematica, 63, 4 (2006), 467–496.
- [14] D. Cioranescu, P. Donato and R. Zaki, Asymptotic behavior of elliptic problems in perforated domains with nonlinear boundary conditions, Asymptotic Analysis, 53, 4 (2007), 209–235.
- [15] D. Cioranescu and J. Saint Jean Paulin, Homogenization in open sets with holes, J. Mathematical Analysis and Applications, 71 (1979), 590–607.
- [16] D. Cioranescu and J. Saint Jean Paulin, *Homogenization of Reticulated Structures*, Applied Mathematical Sciences 136, Springer-Verlag New York, 1999.
- [17] A. Damlamian, From linear to nonlinear by the homogenizations of cracks, in *Proceedings of the conference in honor of Mitsuharu Ôtani, Tokyo, October 2009*, to appear in Math. Sci. Appl. Gakuto Internat. Ser. Math. Sci. Appl., Gakkotosho, Tokyo, 2010.
- [18] A. Damlamian and P. Donato, Which sequences of holes are admissible for periodic homogenization with Neumann boundary condition?, ESAIM : Control, Optimization and Calculus of Variations. Special Volume: A Tribute to Jacques-Louis Lions, 8 (2002), 555–585.

- [19] P. Donato, A. Gaudiello and L. Sgambati, Homogenization of bounded solutions of elliptic equations with quadratic growth in periodically perforated domains, *Asymptotic Analysis*, 16 (1998), 223–243.
- [20] M. Ghergu, G. Griso, H. Mechkour, B. Miara, Homogenization of thin piezoelectric perforated shells, *M2AN Math. Model. Numer. Anal.* 41 (2007), 5, 875–895.
- [21] G. Griso, Error estimate and unfolding for periodic homogenization, *Asymptot. Anal.*, 40 (2004), 269–286.
- [22] G. Griso and M. T. Sanchez, Homogenization of an elasticity problem for a catalyst support by using the unfolding method, *Asymptot. Anal.*, 67 (2010), 45–84.
- [23] W. Jäger and A. Mikelić, Homogenization of the Laplace equation in a partially perforated domain, in *Homogenization*, Ser. Adv. Math. Appl. Sci., 50, World Sci. Publ., 1999.
- [24] M. Neuss-Radu, Some extensions of two-scale convergence, *C. R. Acad. Sci. Paris, Série 1*, 322 (1996), 899–904.
- [25] D. Onofrei, The unfolding operator near a hyperplane and its applications to the Neumann sieve model, *Adv. Math. Sci. Appl.* 16 (2006), no. 1, 239–258.